Modular representation theory of finite groups

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Course notes

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Preface

Throughout these notes, G will always be a finite group and p a prime. The aim of the course will be to introduce the audience to the modular representation theory of finite groups, introduced by Brauer in the 1930s.

These notes are based on Gabriel Navarro's book [**Nav98**] and most of the notation and proofs are inherited from there. My notation for finite groups follows [**Isa08**] and for complex characters [**Nav18**] or [**Isa06**]. Certain notations (such as the conjugacy class sums) appeared first in Britta Späth's papers.

Please, let me know of any mistakes at josep.m.martinez@uv.es. This version is from October 9th, 2024.

Contents

LECTURE 1

Brauer characters

1.1. Modules and representations

In these notes we avoid modules as much as possible and only invoke them whenever necessary. If F is a field (and very soon, F will be a very particular field) then FG denotes the group algebra of G with coefficients in F. If $\mathrm{char}(F)$ divides $|G|$ then Maschke's theorem (and therefore Wedderburn's) no longer applies and things get a bit more complicated, but also more interesting.

An F-**representation** of G is a group homomorphism $\mathcal{X} : G \to \mathrm{GL}_n(F)$. By extending linearly, we may view these as algebra homomorphisms $\mathcal{X} : FG \to \mathrm{Mat}_n(F)$, also known as representations of FG. By restricting, representations of FG give F-representations of G.

If V is an FG-module, then V induces a representation of FG by choosing a basis $\mathcal{B}$ of V and defining $\mathcal{X}(x)$ to be the matrix associated to $v \mapsto vx$ with respect to $\mathcal{B}$. Conversely, if $V = F^n$ and $\mathcal{X}$ is a representation of FG, then V becomes an FG-module by defining $vx = v\mathcal{X}(x)$ for $x \in FG$. Therefore the study of FG-modules is equivalent to the study of representations of FG (and therefore to the study of F-representations of G).

Two representations $\mathcal{X}_1, \mathcal{X}_2$ of FG are **similar** if there is a regular matrix $M \in \mathrm{GL}_n(F)$ with $M^{-1}\mathcal{X}_1(x)M = \mathcal{X}_2(x)$ for all $x \in FG$. It is straightforward to check that $\mathcal{X}_1$ and $\mathcal{X}_2$ are similar if and only if their associated FG-modules are isomorphic.

We say a representation is **irreducible** if its associated FG-module is simple. If FG is semisimple then it is well known that every representation $\mathcal{X}$ of G is similar to a diagonal representation

$$\begin{pmatrix} \mathcal{X}_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \mathcal{X}_t \end{pmatrix}$$

but this is not the case if FG is not semisimple. However we can still guarantee that $\mathcal{X}$ is similar to a representation of the form

$$\begin{pmatrix} \mathcal{X}_1 & \dots & * \\ \vdots & \ddots & \vdots \\ 0 & \dots & \mathcal{X}_t \end{pmatrix}$$

where the $*$ is not necessarily zero (so the representation is in upper triangular block form). A representation of G is irreducible if and only if it is not similar to a representation in block form

$$\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}.$$

1.2. Brauer characters

Let $\mathbf{R}$ denote the ring of algebraic integers in $\mathbb{C}$. It is well known that complex characters take values in $\mathbf{R}$. Let M be a maximal ideal of $\mathbf{R}$ containing $p\mathbf{R}$. Then $F := \mathbf{R}/M$ is a field of characteristic p and let

$$^* : \mathbf{R} \to F$$

be the canonical ring epimorphism. Let

$$\mathbf{U} = \{\xi \in \mathbb{C} \mid \xi^m = 1 \text{ for some integer } m \text{ coprime to } p\}.$$

Notice that $\mathbb{Z}^* = \mathbb{Z}/p\mathbb{Z}$.

Lemma 1.1. *The following hold.*

(i) *The restriction* $^* : \mathbf{U} \to F^\times$ *is a group isomorphism.*

(ii) F *is the algebraic closure of* $\mathbb{Z}/p\mathbb{Z}$*.*

Let $G^0 = \{x \in G \mid p \nmid o(x)\}$ (warning: this is not a subgroup of G in general!), and let $\mathcal{X} : G \to \mathrm{GL}_n(F)$ be an F-representation. Let $g \in G^0$. Since F is algebraically closed and g has finite order, $\mathcal{X}(g)$ is diagonalizable and its eigenvalues lie in $F^\times$. By Lemma 1.1, there exist uniquely determined $\xi_1, \dots, \xi_n \in \mathbf{U}$ such that $\mathcal{X}(g)$ is similar to $\mathrm{diag}(\xi_1^*, \dots, \xi_n^*)$. Then the map

$$\begin{aligned} \varphi : G^0 &\to \mathbb{C} \\ g &\mapsto \xi_1 + \dots + \xi_n \end{aligned}$$

is the **Brauer character** afforded by $\mathcal{X}$. We denote by $\mathrm{IBr}(G)$ the set of irreducible Brauer characters (associated to irreducible F-representations). The set $\mathrm{IBr}(G)$ may depend on the maximal ideal M chosen.

Of course, the restriction that $g \in G^0$ above is unnecessary to find such $\xi_1, \dots, \xi_n \in \mathbf{U}$. We sill justify this restriction below.

The following is totally straightforward and can be proved in the exact same way as for complex characters. We denote by $\overline{x}$ the complex conjugate of $x \in \mathbb{C}$.

LEMMA 1.2. *Let φ be a Brauer character of G. Then*

(i) $\varphi \in \mathrm{cf}(G^0)$,

(ii) $\varphi(g^{-1}) = \overline{\varphi(g)}$,

(iii) $\overline{\varphi} : G^0 \to \mathbb{C}$ *defined by* $\overline{\varphi}(g) := \overline{\varphi(g)}$ *is a Brauer character of G (and its associated module is the dual of the module associated to φ),*

(iv) $H \leqslant G$ *then the restriction* $\varphi_H : H^0 \to \mathbb{C}$ *is a Brauer character of H (and its associated module is the module associated to φ but viewed as an FH-module).*

LEMMA 1.3. *A class function $\psi \in \mathrm{cf}(G^0)$ is a Brauer character iff it is a nonzero nonnegative integral linear combination of irreducible Brauer characters.*

SKETCH OF PROOF. The if direction is immediate, and the only if direction follows by writing the F-representation affording ψ in upper diagonal block form and noticing that ψ is the sum of the Brauer characters appearing in the diagonal. □

We can now justify why we restricted ourselves to G^0. We'll use the following elementary group theoretical fact quite often: every element $g \in G$ can be written as $g = g_p g_{p'}$ where g_p has p-power order, $g_{p'}$ has order coprime to p and $g_p g_{p'} = g_{p'} g_p$. Further, both belong to $\langle g \rangle$.

LEMMA 1.4. *If $\mathcal{X} : G \to \mathrm{GL}_n(F)$ is an F-representation affording the Brauer character φ, then for all $g \in G$ we have*

$$\mathrm{trace}(\mathcal{X}(g)) = \varphi(g_{p'})^*.$$

PROOF. There is no loss in assuming that φ is irreducible and that $G = \langle g \rangle$. Therefore $\mathcal{X} : G \to F^\times$ is a group homomorphism and $\mathcal{X}(g) = \mathcal{X}(g_p)\mathcal{X}(g_{p'})$. Now $\mathcal{X}(g_p)$ has p-power order in $F^\times$ so $\mathcal{X}(g_p) = 1$ and the result follows. □

PROPOSITION 1.5. *The set* $\mathrm{IBr}(G)$ *is linearly independent.*

SKETCH OF PROOF. Use the fact that the set of trace functions of representations $G \to \mathrm{GL}_n(F)$ is linearly independent (see [**Nav98**, Theorem 1.19]) and Lemma 1.4. □

DEFINITION 1.6. *If $\chi \in \mathrm{Char}(G)$ then χ^0 denotes the restriction of χ to G^0.*

More notation. We let $\mathcal{S}$ be the localization of $\mathbf{R}$ at M, that is

$$\mathcal{S} = \{r/s \mid r \in \mathbf{R}, s \in \mathbf{R}\backslash M\}$$

and extend * to an homomorphism $^* : \mathcal{S} \to F$ by

$$(r/s)^* = r^*(s^*)^{-1}.$$

Perhaps it is useful to mention here that an integer belongs to M if and only if it is divisible by p.

The following deep theorem is [**Nav98**, Theorem 2.7].

THEOREM 1.7. *If $\mathcal{X} : G \to \mathrm{GL}_n(\mathbb{C})$ is a complex representation, then there is a representation $\mathcal{Y}$ of G similar to $\mathcal{X}$ with entries in $\mathcal{S}$.*

We extend * to a ring homomorphism $\mathrm{Mat}_n(\mathcal{S}) \to \mathrm{Mat}_n(F)$ by applying * to the entries of a matrix $A \in \mathrm{Mat}_n(\mathcal{S})$. Notice that $\det(A^*) = \det(A)^*$. We also extend it to a ring homomorphism $\mathcal{S}[x] \to F[x]$. It is straightforward to check that if a polynomial $p(x) \in \mathcal{S}[x]$ has all roots $\alpha_1, \dots, \alpha_t$ in $\mathcal{S}$ then $p(x)^*$ has roots $\alpha_1^*, \dots, \alpha_t^*$.

LEMMA 1.8. *If $\mathcal{X} : G \to \mathrm{GL}_n(\mathbb{C})$ affords χ with entries in $\mathcal{S}$. Then $\mathcal{X}^* : G \to \mathrm{GL}_n(F)$ defined by*

$$\mathcal{X}^*(g) := \mathcal{X}(g)^*$$

affords the Brauer character χ^0.

PROOF. First notice that $\mathcal{X}^* : G \to \mathrm{GL}_n(F)$ is in fact an homomorphism (we are applying * to all the entries). Let $g \in G^0$ and let $\xi_1, \dots, \xi_t \in \mathbf{U}$ be the eigenvalues of $\mathcal{X}(g)$. Using that $\det(xI - \mathcal{X}(g))^* = \det(xI - \mathcal{X}^*(g))$ we have that $\xi_1^*, \dots, \xi_t^*$ are the eigenvalues of $\mathcal{X}^*(g)$. □

It is true (but much harder and not necessary for our purposes) that we may find a represenetation with entries in $\mathbf{R}$ affording any $\chi \in \mathrm{Char}(G)$. However, it will be convenient to work in $\mathcal{S}$.

It follows from Lemma 1.8 and Theorem 1.7 that for all $\chi \in \mathrm{Char}(G)$ we have that χ^0 is a Brauer character of G.

1.3. Decomposition numbers

Since $\mathrm{IBr}(G)$ is a linearly independent set of $\mathbb{C}$, it follows that we may write

$$\chi^0 = \sum_{\varphi \in \mathrm{IBr}(G)} d_{\chi\varphi}\varphi$$

for certain uniquely defined integers (which are nonnegative by 1.3). The numbers $d_{\chi\varphi}$ are called the **decomposition numbers**.

THEOREM 1.9. *The set* $\mathrm{IBr}(G)$ *is a basis of* $\mathrm{cf}(G^0)$*. In particular,* $|\mathrm{IBr}(G)|$ *coincides with the number of conjugacy classes of* p*-regular elements of* G*.*

PROOF. By Proposition 1.5 it suffices to show that any $\beta \in \mathrm{cf}(G^0)$ can be written as a linear combination of $\mathrm{IBr}(G)$. Now, let $\delta \in \mathrm{cf}(G)$ be any extension of β (so that $\delta^0 = \beta$). Then using that $\mathrm{Irr}(G)$ is a basis of $\mathrm{cf}(G)$ we may write

$$\delta = \sum_{\chi \in \mathrm{Irr}(G)} a_\chi \chi$$

and therefore

$$\beta = \sum_{\chi \in \mathrm{Irr}(G)} a_\chi \chi^0 = \sum_{\chi \in \mathrm{Irr}(G)} a_\chi \left(\sum_{\varphi \in \mathrm{IBr}(G)} d_{\chi\varphi} \varphi \right)$$

and we are done. □

We define the **decomposition matrix** of G by

$$D = (d_{\chi\varphi})_{\chi \in \mathrm{Irr}(G), \varphi \in \mathrm{IBr}(G)}.$$

(That is, Brauer characters in the columns, ordinary characters in rows.) This matrix is completely independent of the maximal ideal M chosen.

PROBLEM 1.10. *Prove that the decomposition matrix has maximum rank.*

As a consequence, for any $\varphi \in \mathrm{IBr}(G)$ there is some $\chi \in \mathrm{Irr}(G)$ with $d_{\chi\varphi} \neq 0$.

PROBLEM 1.11. *If* p *does not divide* $|G|$*, prove that* $\mathrm{IBr}(G) = \mathrm{Irr}(G)$*.*

(Hint: use Maschke and Wedderburn to obtain $\sum_{\varphi \in \mathrm{IBr}(G)} \varphi(1)^2 = |G|$.)

The **Cartan matrix** is defined by $C = D^t D$, and it has many interesting and deep properties which, unfortunately, we shall not discuss.

1.4. Projective indecomposable characters

Let $\varphi \in \mathrm{IBr}(G)$. We define the **projective indecomposable character** associated to φ by

$$\Phi_\varphi := \sum_{\chi \in \mathrm{Irr}(G)} d_{\chi\varphi} \chi.$$

Here we will only prove the necessary facts for our purposes, but these characters are certainly very interesting. They are a basis of the set of class functions of G which vanish off the p-regular conjugacy classes, and they have some properties connected to the Cartan matrix. For more on these characters, see [**Nav98**, Chapter 2].

PROPOSITION 1.12. *Let* $\varphi \in \mathrm{IBr}(G)$ *and* $g \in G$ *with* $p \mid o(g)$*. Then* $\Phi_\varphi(g) = 0$*.*

PROOF. Let $x \in G^0$. By the second orthogonality relation for complex characters, we have that

$$\sum_{\chi \in \mathrm{Irr}(G)} \overline{\chi(g)}\chi(x) = 0$$

but the LHS equals

$$\sum_{\chi \in \mathrm{Irr}(G)} \overline{\chi(g)} \left(\sum_{\varphi \in \mathrm{IBr}(G)} d_{\chi\varphi}\varphi(x) \right) = \sum_{\varphi \in \mathrm{IBr}(G)} \varphi(x) \left(\sum_{\chi \in \mathrm{Irr}(G)} d_{\chi\varphi}\overline{\chi(g)} \right)$$
$$= \sum_{\varphi \in \mathrm{IBr}(G)} \overline{\Phi_\varphi(g)}\varphi(x)$$

which implies that the linear combination

$$\sum_{\varphi \in \mathrm{IBr}(G)} \overline{\Phi_\varphi(g)}\varphi = 0$$

and since $\mathrm{IBr}(G)$ is a linearly independent set, this implies $\Phi_\varphi(g) = 0$ for p-singular g. □

COROLLARY 1.13 (Dickson). *If $\varphi \in \mathrm{IBr}(G)$ then $|G|_p$ divides $\Phi_\varphi(1)$.*

PROOF. Notice that Φ_φ is a character of G. Then if $P \in \mathrm{Syl}_p(G)$ we have $(\Phi_\varphi)_P$ is a character of P and therefore $[(\Phi_\varphi)_P, 1_P]$ is a nonnegative integer. Now Φ_φ vanishes in every element of P except 1, which means that

$$[(\Phi_\varphi)_P, 1_P] = \frac{1}{|P|}\sum_{x \in P} \Phi_\varphi(x) = \frac{\Phi_\varphi(1)}{|P|}$$

and we are done. □

1.5. Kernels of Brauer characters

Let $\varphi \in \mathrm{IBr}(G)$ be afforded by the F-representation $\mathcal{X}$. Then we define

$$\ker(\varphi) := \ker(\mathcal{X}).$$

Notice that, unlike with ordinary characters, we cannot guarantee $\ker(\varphi)$ is the set of elements $g \in G$ where $\varphi(g) = \varphi(1)$ since $\varphi(g)$ is only defined for p-regular elements g. (It is true that if $g \in G^0$ then $g \in \ker(\varphi)$ if and only if $\varphi(g) = \varphi(1)$, see [**Nav98**, Lemma 6.11].)

If $\varphi \in \mathrm{IBr}(G)$ with $N \subseteq \ker(\varphi)$ then we can define $\overline{\varphi}(Ng) = \varphi(g_{p'})$ and $\overline{\varphi} \in \mathrm{IBr}(G/N)$ (check this! Can you find an F-representation that affords $\overline{\varphi}$?). We identify $\overline{\varphi}$ with φ and thus view $\mathrm{IBr}(G/N)$ as a subset of $\mathrm{IBr}(G)$.

THEOREM 1.14. *Let $\mathcal{X}$ be an irreducible F-representation of G. Then $\mathbf{O}_p(G) \subseteq \ker(\mathcal{X})$.*

Proof. Let $P = \mathbf{O}_p(G)$. We have that FP has a unique simple module, the trivial one. Let V be a (simple) FG-module affording $\mathcal{X}$. Viewing V as an FP-module we have that if $0 < W \leqslant V$ is a simple FP-submodule, $W \subseteq \mathbf{C}_V(P)$.

Now using that $P \lhd G$ it is straightforward to see that $\mathbf{C}_V(P)$ is G-invariant and thus it is an FG-submodule of V. Since V is simple it follows that $\mathbf{C}_V(P) = V$, or in other words that $vx = v$ for all $v \in V$ and $x \in P$. Therefore, for $x \in P$ we have that $\mathcal{X}(x) = I$ and we are done. □

Problem 1.15. *A Brauer character φ is said to be **linear** if $\varphi(1) = 1$. Denote by* $\mathrm{LinBr}(G)$ *the set of linear Brauer characters of G. Prove that*

(i) $\mathrm{LinBr}(G) \subseteq \mathrm{IBr}(G)$,

(ii) *if $N = \mathbf{O}^{p'}(G)G'$ (the smallest normal subgroup N such that G/N is abelian and of p'-order), $\chi \mapsto \chi^0$ is a bijection* $\mathrm{Irr}(G/N) \to \mathrm{LinBr}(G)$,

(iii) $|\mathrm{LinBr}(G)| = |G : G'|_{p'}$,

(iv) $\mathrm{LinBr}(G)$ *is a finite group,*

(v) *the map from (ii) is a group isomorphism.*

LECTURE 2

Blocks

If $x \in G$ then we denote by $\mathfrak{Cl}_G(x)$ the conjugacy class of G containing x. Then

$$\mathfrak{Cl}_G(x)^+ = \sum_{y \in \mathfrak{Cl}_G(x)} y \in \mathbf{Z}(KG)$$

for any field K, and in fact we have that $\{\mathfrak{Cl}_G(x)^+ \mid x \in G/\sim\}$ is a basis of $\mathbf{Z}(KG)$ (we denote by $G/\sim$ a set of representatives of the conjugacy classes of G).

It is well known that if $\chi \in \mathrm{Irr}(G)$ then $\chi(x) \in \mathbf{R}$ for all $x \in G$, and in fact

$$\frac{|\mathfrak{Cl}_G(x)|\chi(x)}{\chi(1)} \in \mathbf{R}.$$

Thus, χ defines an algebra homomorphism

$$\omega_\chi : \mathbf{Z}(\mathbb{C}G) \to \mathbb{C}$$

by setting

$$\omega_\chi(\mathfrak{Cl}_G(x)^+) = \frac{|\mathfrak{Cl}_G(x)|\chi(x)}{\chi(1)} \in \mathbf{R}.$$

In fact, if $\mathcal{X}$ is a complex representation affording χ, we have that $\mathcal{X}(\mathfrak{Cl}_G(x)^+) = \omega_\chi(\mathfrak{Cl}_G(x)^+)I_n$.

Using the fact that $\omega_\chi(\mathfrak{Cl}_G(x)^+) \in \mathbf{R}$ we may construct an F-linear map

$$\begin{aligned} \lambda_\chi : \mathbf{Z}(FG) &\to F \\ \mathfrak{Cl}_G(x)^+ &\mapsto \omega_\chi(\mathfrak{Cl}_G(x)^+)^* \end{aligned}$$

and extending linearly.

Let $\varphi \in \mathrm{IBr}(G)$ be afforded by an F-representation $\mathcal{X}$. Notice that $\mathcal{X}(\mathfrak{Cl}_G(x)^+)$ is a scalar matrix, so we may write

$$\mathcal{X}(\mathfrak{Cl}_G(x)^+) = \lambda_\varphi(\mathfrak{Cl}_G(x)^+)I_n$$

and again, this defines an F-linear map $\lambda_\varphi : \mathbf{Z}(FG) \to F$.

Definition 2.1. *The p-**blocks** of G are the equivalence classes in* $\mathrm{Irr}(G) \cup \mathrm{IBr}(G)$ *under the relation $\chi \sim \varphi$ if $\lambda_\chi = \lambda_\varphi$.*

If B is a p-block then $\mathrm{Irr}(B) = B \cap \mathrm{Irr}(G)$ and $\mathrm{IBr}(B) = B \cap \mathrm{IBr}(G)$. You will have to believe me for the moment, but these do not depend on the choice of the maximal ideal M (we'll see why later, maybe). Also, it makes sense to set the notation $\lambda_B := \lambda_\chi$ for whatever $\chi \in \mathrm{Irr}(B) \cup \mathrm{IBr}(B)$. We denote by $\mathrm{bl}(\psi)$ the block to which some $\psi \in \mathrm{Irr}(G) \cup \mathrm{IBr}(G)$ belongs.

THEOREM 2.2. *Let* $\chi \in \mathrm{Irr}(G)$ *and* $\varphi \in \mathrm{IBr}(G)$ *be such that* $d_{\chi\varphi} \neq 0$. *Then* $\lambda_\chi = \lambda_\varphi$.

PROOF. Let $\mathcal{X}$ be a representation taking values in $\mathcal{S}$ that affords χ. Then $\mathcal{X}^*$ is an F-representation affording χ^0. Now $\mathcal{X}^*$ is similar to an F-representation $\mathcal{X}'$ in upper-triangular block form.

Since $d_{\chi\varphi} \neq 0$ then one of the representations $\mathcal{Y}$ appearing in the block diagonal of $\mathcal{X}'$ affords φ. Now for all $x \in G$, $\mathcal{X}'(\mathfrak{Cl}_G(x)^+) = \lambda_\chi(\mathfrak{Cl}_G(x)^+)I_{\chi(1)}$, which implies that $\mathcal{Y}(\mathfrak{Cl}_G(x)^+) = \lambda_\chi(\mathfrak{Cl}_G(x)^+)I_{\varphi(1)}$ as desired. □

It follows from the above that

$$\mathrm{IBr}(B) = \{\varphi \in \mathrm{IBr}(G) \mid d_{\chi\varphi} \neq 0 \text{ for some } \chi \in \mathrm{Irr}(B)\}.$$

Notice also that it implies that, after rearranging *by blocks*, the decomposition matrix has block diagonal form:

$$D = \begin{pmatrix} D_{B_1} & 0 & \dots & 0 \\ 0 & D_{B_2} & \dots & 0 \\ 0 & \dots & \ddots & 0 \\ 0 & \dots & 0 & D_{B_t} \end{pmatrix}.$$

Since D has rank $|\mathrm{IBr}(G)|$, notice that every submatrix D_B has rank $|\mathrm{IBr}(B)|$. In particular, $l(B) := |\mathrm{IBr}(B)| \leqslant |\,\mathrm{Irr}(B)| =: k(B)$. We might see that $k(B) = l(B)$ actually implies $k(B) = 1$. The set of p-blocks of G is denoted by $\mathrm{Bl}(G)$.

2.1. Decomposition matrices for blocks

Our next goal is to show that if $B \in \mathrm{Bl}(G)$ then D_B is not a block diagonal matrix (independently of any rearrangements of rows and columns).

2.1.1. The Brauer graph. We define a graph on $\mathrm{Irr}(G)$ as follows: we link χ and ψ if there is $\varphi \in \mathrm{IBr}(G)$ with $d_{\chi\varphi} \neq 0 \neq d_{\psi\varphi}$. The graph containing $\mathrm{Irr}(G)$ as vertices and with connected vertices the linked characters is known as the **Brauer graph**. We know that any $B \in \mathrm{Bl}(G)$ satisfies that $\mathrm{Irr}(B)$ is the union of connected components of this graph. Our next goal is to see that the connected components of this graph are precisely the sets $\mathrm{Irr}(B)$ with $B \in \mathrm{Bl}(G)$. If $\mathcal{A} \subseteq \mathrm{Irr}(G)$ is the union of connected components of the Brauer graph, then we denote by

$$\mathrm{IBr}(\mathcal{A}) = \{\varphi \in \mathrm{IBr}(G) \mid d_{\chi\varphi} \neq 0 \text{ for some } \chi \in \mathcal{A}\}.$$

PROPOSITION 2.3 (Osima). *Let $\mathcal{A}$ as above. If $g \in G^0$ and $x \in G$ then*

$$\sum_{\chi \in \mathcal{A}} \chi(g)\chi(x) = \sum_{\varphi \in \mathrm{IBr}(\mathcal{A})} \varphi(g)\Phi_\varphi(x)$$

PROOF. First notice that $\varphi \in \mathrm{IBr}(\mathcal{A})$ and $\chi \in \mathrm{Irr}(G)$ then $d_{\chi\varphi} \neq 0$ implies $\chi \in \mathcal{A}$ (indeed, since $\varphi \in \mathrm{IBr}(\mathcal{A})$ there is $\psi \in \mathcal{A}$ with $d_{\psi\varphi} \neq 0$, so χ and ψ are connected, but since $\mathcal{A}$ is a union of connected components we have $\chi \in \mathcal{A}$. It follows that

$$\sum_{\chi \in \mathcal{A}} d_{\chi\varphi}\chi = \Phi_\varphi.$$

Thus

$$\sum_{\chi \in \mathcal{A}} \chi(g)\chi(x) = \sum_{\chi \in \mathcal{A}} \left(\sum_{\varphi \in \mathrm{IBr}(G)} d_{\chi\varphi}\varphi(g) \right) \chi(x) =$$

$$= \sum_{\varphi \in \mathrm{IBr}(\mathcal{A})} \left(\sum_{\chi \in \mathcal{A}} d_{\chi\varphi}\chi(x) \right) \varphi(g) = \sum_{\varphi \in \mathrm{IBr}(\mathcal{A})} \Phi_\varphi(x)\varphi(g)$$

as desired. □

COROLLARY 2.4 (Weak block orthogonality). *Let $B \in \mathrm{Bl}(G)$, $g \in G^0$ and $x \in G \backslash G^0$. Then*

$$\sum_{\chi \in \mathrm{Irr}(B)} \chi(g)\overline{\chi(x)} = 0.$$

PROOF. Apply Proposition 2.3 to $\mathrm{Irr}(B) = \mathcal{A}$ and use Proposition 1.12 to get that the RHS vanishes. □

2.1.2. The primitive central idempotents of $\mathbb{C}G$. We say an element e in an algebra A is an **idempotent** of A if $e^2 = e$ and $e \neq 0$. Further, we say e is a **primitive central idempotent** if it can not be written as a sum of central idempotents. It is a classical fact that if e is a central idempotent in A then e is primitive if and only if eA is an indecomposable (two-sided) ideal of A (i.e. it can not be written as a direct sum of proper ideals).

If we want to decompose FG as a direct sum of indecomposable FG-modules, it might be convenient to find the primitive central idempotents of FG. We have mentioned before that the primitive central idempotents of $\mathbb{C}G$ are given by

$$e_\chi = \frac{\chi(1)}{|G|} \sum_{g \in G} \chi(g^{-1})g$$

where χ runs over $\mathrm{Irr}(G)$, and that $e_\chi \mathbb{C}G$ is the indecomposable two-sided ideal of $\mathbb{C}G$ corresponding to χ. We will use the e_χ's to build the primitive central idempotents of FG in Section 2.3.

If $\psi \in \mathrm{Irr}(G)$ then notice that

$$e_\psi = \frac{\psi(1)}{|G|} \sum_{x \in G/\sim} \psi(x^{-1}) \mathfrak{Cl}_G(x)^+.$$

Thus if $\chi \in \mathrm{Irr}(G)$ we have that, by orthogonality of characters,

$$\omega_\chi(e_\psi) = \frac{\psi(1)}{|G|\chi(1)} \sum_{x \in G/\sim} \psi(x^{-1})\chi(x)|\mathfrak{Cl}_G(x)| = \frac{\psi(1)}{\chi(1)}[\psi, \chi] = \delta_{\chi\psi}.$$

If $\mathcal{A} \subseteq \mathrm{Irr}(G)$ we will denote by

$$f_\mathcal{A} := \sum_{\chi \in \mathcal{A}} e_\chi.$$

Since $e_\chi \in \mathbf{Z}(\mathbb{C}G)$ it follows that we may write

$$f_\mathcal{A} = \sum_{x \in G/\sim} f_\mathcal{A}(\mathfrak{Cl}_G(x)^+)\mathfrak{Cl}_G(x)^+.$$

Let us now work out a formula for the coefficient $f_\mathcal{A}(\mathfrak{Cl}_G(x)^+)$. By using the formula above, we have

$$\sum_{\chi \in \mathcal{A}} e_\chi = \sum_{\chi \in \mathcal{A}} \frac{1}{|G|} \left(\sum_{x \in G/\sim} \chi(1)\chi(x^{-1})\mathfrak{Cl}_G(x)^+ \right) =$$
$$= \frac{1}{|G|} \sum_{x \in G/\sim} \left(\sum_{\chi \in \mathcal{A}} \chi(1)\chi(x^{-1}) \right) \mathfrak{Cl}_G(x)^+$$

and we conclude that

$$f_\mathcal{A}(\mathfrak{Cl}_G(x)^+) = \frac{1}{|G|} \sum_{\chi \in \mathcal{A}} \chi(1)\chi(x^{-1})$$

and fortunately, Proposition 2.3 gives an alternative description for these coefficients!

Proposition 2.5. *Let $\mathcal{A} \subseteq \mathrm{Irr}(G)$ be a union of connected components of the Brauer graph. Then*

(i) $f_\mathcal{A} \in \mathbf{Z}(\mathcal{S}G)$ *(that is, $f_\mathcal{A}(\mathfrak{Cl}_G(x)^+) \in \mathcal{S}$),*

(ii) $f_\mathcal{A}(\mathfrak{Cl}_G(x)^+) = 0$ *if $x \notin G^0$.*

Proof. Let $x \in G \backslash G^0$. By Proposition 2.3, we have

$$f_\mathcal{A}(\mathfrak{Cl}_G(x)^+) = \frac{1}{|G|} \sum_{\varphi \in \mathrm{IBr}(\mathcal{A})} \varphi(1)\Phi_\varphi(x^{-1}).$$

Now since x^{-1} is not p-regular and Φ_φ vanishes outside G^0, we conclude that $f_\mathcal{A}(\mathfrak{Cl}_G(x)^+) = 0$, and part (ii) follows.

If $x \in G^0$ then by the same result but reversing the role of x^{-1} and 1 we get

$$f_{\mathcal{A}}(\mathfrak{Cl}_G(x)^+) = \frac{1}{|G|} \sum_{\varphi \in \mathrm{IBr}(\mathcal{A})} \varphi(x^{-1})\Phi_\varphi(1)$$

and since we know that $\varphi(x^{-1})$ is a sum of elements in $\mathbf{U} \subseteq \mathbf{R}$ then it suffices to show that

$$\frac{\Phi_\varphi(1)}{|G|} \in \mathcal{S}$$

but this holds because $|G|_p$ divides $\Phi_\varphi(1)$ (since $M \cap \mathbb{Z} = p\mathbb{Z}$, an integer belongs to M if and only if it is divisible by p). □

We extend our homomorphism $^* : \mathcal{S} \to F$ even more to an homomorphism $\mathcal{S}G \to FG$ by

$$\left(\sum_{x \in G} a_x x \right)^* = \sum_{x \in G} a_x^* x$$

and notice that it maps $\mathbf{Z}(\mathcal{S}G)$ onto $\mathbf{Z}(FG)$ by

$$\left(\sum_{x \in G/\sim} a_x \mathfrak{Cl}_G(x)^+ \right)^* = \sum_{x \in G/\sim} a_x^* \mathfrak{Cl}_G(x)^+.$$

If $z \in \mathbf{Z}(\mathcal{S}G)$ and $\chi \in \mathrm{Irr}(G)$ then $\omega_\chi(z)^* = \lambda_\chi(z^*)$.

Finally, we get to the main result of this section.

Theorem 2.6. *If* $\mathcal{A} \subseteq \mathrm{Irr}(G)$ *is such that* $f_{\mathcal{A}} \in \mathbf{Z}(\mathcal{S}G)$ *then there is* $\Omega \subseteq \mathrm{Bl}(G)$ *with*

$$\mathcal{A} = \bigcup_{B \in \Omega} \mathrm{Irr}(B).$$

In other words, if $\chi \in \mathcal{A}$ *then* $\mathrm{Irr}(\mathrm{bl}(\chi)) \subseteq \mathcal{A}$.

Proof. If $\chi \in \mathrm{Irr}(G)$, then have that $\omega_\chi(f_{\mathcal{A}}) \neq 0$ if and only if $\chi \notin \mathcal{A}$ (and $\omega_\chi(f_{\mathcal{A}}) = 1$ if $\chi \in \mathcal{A}$). Since $f_{\mathcal{A}} \in \mathbf{Z}(\mathcal{S}G)$, by the above discussion this implies that $\lambda_\chi(f_{\mathcal{A}}^*) = 0$ if $\chi \notin \mathcal{A}$ and $\lambda_\chi(f_{\mathcal{A}}^*) = 1$ otherwise. Since $\lambda_\chi = \lambda_\psi$ if ψ and χ belong to the same block, it follows that $\mathcal{A}$ contains every ordinary character in the block of χ or it contains none. □

Corollary 2.7. *If* $B \in \mathrm{Bl}(G)$ *then* $\mathrm{Irr}(B)$ *is a single connected component of the Brauer graph. In particular,* D_B *is not of the form*

$$\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}.$$

Proof. Assume $\mathcal{A} \subseteq \mathrm{Irr}(B)$ is a connected component of the Brauer graph. By Proposition 2.5, $f_{\mathcal{A}} \in \mathbf{Z}(\mathcal{S}G)$ so $\mathcal{A}$ contains all of $\mathrm{Irr}(B)$ and the first part follows. For the second, notice that such a form of the decomposition matrix would imply that there are two distinct connected components inside $\mathrm{Irr}(B)$. □

2.2. Blocks of defect zero

The following results were proved by Brauer and Nesbitt, and can be deduced from the results in the previous section.

Theorem 2.8 (Brauer–Nesbitt). *Let $B \in \mathrm{Bl}(G)$. Then following are equivalent:*

(i) $k(B) = l(B)$,

(ii) *if* $\chi \in \mathrm{Irr}(B)$ *and all* $g \in G \backslash G^0$ *we have* $\chi(g) = 0$,

(iii) *there is* $\chi \in \mathrm{Irr}(B)$ *with* $\chi(1)_p = |G|_p$,

(iv) $k(B) = 1$.

Proof. If $k(B) = l(B)$, then D_B is a square matrix of maximal rank and therefore it is invertible. Write $(D_B)^{-1} = (a_{\varphi\chi})$ and for a fixed $\chi \in \mathrm{Irr}(B)$ compute

$$\sum_{\varphi \in \mathrm{IBr}(B)} a_{\varphi\chi}\Phi_\varphi = \sum_{\varphi \in \mathrm{IBr}(B)} a_{\varphi\chi}\left(\sum_{\psi \in \mathrm{Irr}(B)} d_{\psi\varphi}\psi\right) = \sum_{\psi \in \mathrm{Irr}(B)}\left(\sum_{\varphi \in \mathrm{IBr}(B)} a_{\varphi\chi} d_{\psi\varphi}\right)\psi$$

And notice that

$$\sum_{\varphi \in \mathrm{IBr}(B)} a_{\varphi\chi} d_{\psi\varphi}$$

is precisely the value entry of the matrix $D_B(D_B)^{-1}$ in the coordinate corresponding to $\chi\psi$, i.e. $\delta_{\chi\psi}$. Therefore

$$\sum_{\varphi \in \mathrm{IBr}(B)} a_{\varphi\chi}\Phi_\varphi = \chi$$

and it follows that χ vanishes in the p-singular elements, so (i) implies (ii).

Now if (ii) holds then $[\chi_P, 1_P] = \frac{1}{|P|}\sum_{x \in P}\chi(x) = \frac{\chi(1)}{|P|}$ must be an integer, so (iii) follows.

If (iii) holds then

$$e_\chi = \frac{\chi(1)}{|G|}\sum_{x \in G}\chi(x^{-1})x$$

belongs to $\mathbf{Z}(\mathcal{S}G)$ so by Theorem 2.6 we have that $\{\chi\} = \mathrm{Irr}(B)$ and (iv) holds.

Since $l(B) > 0$, (iv) implies (i). □

In fact, the following follows too, but this needs a fact we have not proved: that φ is a $\mathbb{Z}$-linear combination of $\{\chi^0 \mid \chi \in \mathrm{Irr}(B)\}$.

Corollary 2.9. *If* $\mathrm{Irr}(B) = \{\chi\}$ *then* $\mathrm{IBr}(B) = \{\chi^0\}$.

This is a phenomenon that can only happen in blocks with a unique Brauer character.

Problem 2.10. *Let* $B \in \mathrm{Bl}(G)$ *with* $l(B) > 1$. *Then there is* $\chi \in \mathrm{Irr}(B)$ *with* $\chi^0 \notin \mathrm{IBr}(B)$.

2.3. The primitive central idempotents of FG

We devote this last section to connecting the module-theoretic point of view of blocks to our character-theoretic one. Let $B \in \mathrm{Bl}(G)$ and recall that we write

$$f_B := f_{\mathrm{Irr}(B)} = \sum_{\chi \in \mathrm{Irr}(B)} e_\chi.$$

Since $f_B \in \mathbf{Z}(\mathcal{S}G)$ we may apply the * homomorphism to obtain an element of $e_B := (f_B)^* \in \mathbf{Z}(FG)$.

Since $1 = \sum_{\chi \in \mathrm{Irr}(G)} e_\chi$ it follows that $1 = \sum_{B \in \mathrm{Bl}(G)} e_B$ (notice that one is the identity in $\mathbb{C}$ and the other one in F).

Recall that the Jacobson radical $\mathbf{J}(A)$ of an F-algebra A is the intersection of all maximal right ideals of A.

Theorem 2.11. *The set of all primitive idempotents of* $\mathbf{Z}(FG)$ *is* $\{e_B \mid B \in \mathrm{Bl}(G)\}$, $\{\lambda_B \mid B \in \mathrm{Bl}(G)\}$ *is the set of all algebra homomorphisms* $\mathbf{Z}(FG) \to F$ *and* $\lambda_B(e_{B'}) = \delta_{BB'}$. *Furthermore*

$$\mathbf{J}(\mathbf{Z}(FG)) = \bigcap_{B \in \mathrm{Bl}(G)} \ker(\lambda_B).$$

Since the e_B's are primitive central idempotents, it follows that FGe_B is an indecomposable two-sided ideal of FG (so it can not be written as a direct sum of proper two-sided ideals). Further, using that $1 = \sum_{B \in \mathrm{Bl}(G)} e_B$ we also get

$$FG = \bigoplus_{B \in \mathrm{Bl}(G)} FGe_B.$$

Many authors write $B = FGe_B$. The element e_B is the identity in FGe_B.

Theorem 2.12. *Let* $B \in \mathrm{Bl}(G)$.

(i) *Let* $\chi \in \mathrm{Irr}(G)$ *be afforded by the* $\mathbb{C}G$*-module* V. *Then* $\chi \in \mathrm{Irr}(B)$ *if and only if* $Vf_B = V$. *Otherwise,* $Vf_B = 0$.

(ii) *Let $\varphi \in \mathrm{IBr}(G)$ be afforded by the FG-module V. Then $\varphi \in \mathrm{IBr}(B)$ if and only if $Ve_B = V$. Otherwise $Ve_B = 0$.*

In summation, we have a decomposition $FG = \bigoplus B$ and to decide whether $\psi \in \mathrm{Irr}(G) \cup \mathrm{IBr}(G)$ belongs to B we check that $\lambda_\psi(e_B) \neq 0$.

LECTURE 3

Defect groups

Let $x \in G$. We denote by $\delta(\mathfrak{Cl}_G(x)) = \{D^g \mid D \in \mathrm{Syl}_p(\mathbf{C}_G(x)), g \in G\}$, and these are called the **defect groups** of the conjugacy class $\mathfrak{Cl}_G(x)$. If Q, P are p-subgroups of G, we write $Q \subseteq_G P$ if Q is contained in a G-conjugate of P.

3.1. The Min–Max theorem

Recall from the previous chapter that if $B \in \mathrm{Bl}(G)$ then we denoted $f_B = \sum_{\chi \in \mathrm{Irr}(B)} e_\chi$ and that $e_B = (f_B)^*$ is the primitive central idempotent generating the block B.

Recall that we wrote

$$f_B = \sum_{x \in G/\sim} f_B(\mathfrak{Cl}_G(x)^+)\mathfrak{Cl}_G(x)^+ \in \mathbf{Z}(\mathbb{C}G).$$

Similarly, we may write

$$e_B = \sum_{x \in G/\sim} a_B(\mathfrak{Cl}_G(x)^+)\mathfrak{Cl}_G(x)^+ \in \mathbf{Z}(FG)$$

where $a_B(\mathfrak{Cl}_G(x)^+) = f_B(\mathfrak{Cl}_G(x)^+)^*$ (recall that these coefficients lie in $\mathcal{S}$ by Proposition 2.5).

We know that if $B, B' \in \mathrm{Bl}(G)$ then $\lambda_B(e_{B'}) = \delta_{BB'}$. Applying this to the previous equality we get that there are some $x \in G/\sim$ such that

$$a_B(\mathfrak{Cl}_G(x)^+)\lambda_B(\mathfrak{Cl}_G(x)^+) \neq 0.$$

A class $\mathfrak{Cl}_G(x)$ satisfying the above condition is known as a **defect class** for B.

Theorem 3.1. *Let $B \in \mathrm{Bl}(G)$ and $x, y \in G$. If*

$$a_B(\mathfrak{Cl}_G(x)^+)\lambda_B(\mathfrak{Cl}_G(x)^+) \neq 0 \neq a_B(\mathfrak{Cl}_G(y)^+)\lambda_B(\mathfrak{Cl}_G(y)^+)$$

then $\delta(\mathfrak{Cl}_G(x)) = \delta(\mathfrak{Cl}_G(y))$.

Definition 3.2. *The **defect groups** of a block B are the defect groups of a defect class.*

We denote by $\delta(B)$ the set of defect groups of B. We write $\mathrm{Bl}(G|D)$ for the set of blocks of G with defect group D.

THEOREM 3.3 (Min–Max). *Let* $B \in \mathrm{Bl}(G)$*,* $D_B \in \delta(B)$*,* $g \in G$ *and* $D_g \in \delta(\mathfrak{Cl}_G(g))$*. The following hold.*

- *If* $\lambda_B(\mathfrak{Cl}_G(g)^+) \neq 0$ *then* $D_B \subseteq_G D_g$*.*
- *If* $a_B(\mathfrak{Cl}_G(g)^+) \neq 0$ *then* $D_g \subseteq_G D_B$*.*

This approach might seem a bit weird and in fact the result is somewhat circular. Defect groups are usually defined in a different way, but such that the Min–Max theorem applies. This ends up proving that the defect groups are well defined by obtaining Theorem 3.1 as a corollary of the Min–Max theorem.

3.2. Numerical defect and height zero characters

Write $|G|_p = p^a$. We define the **numerical defect** of a block $B \in \mathrm{Bl}(G)$ to be the nonnegative integer $\mathrm{d}(B)$ such that

$$p^{a-\mathrm{d}(B)} = \min\{\chi(1)_p \mid \chi \in \mathrm{Irr}(B)\}.$$

It follows that if $|G|_p = p^a$ then for every $\chi \in \mathrm{Irr}(G)$ we may write

$$\chi(1)_p = p^{a-\mathrm{d}(B)+h_\chi}$$

for some nonnegative integer h_χ, called the **height** of χ. The characters $\chi \in \mathrm{Irr}(B)$ such that $\mathrm{d}(B) = \chi(1)_p$ (i.e., $h_\chi = 0$) are called **height zero** characters. We denote by $\mathrm{Irr}_0(B)$ the set of height zero characters of B.

Our next goal is to prove the following

THEOREM 3.4. *Let* $B \in \mathrm{Bl}(G)$ *and* $D \in \delta(B)$*. Then* $|D| = p^{\mathrm{d}(B)}$*.*

We need some previous results, which we will not prove.

PROPOSITION 3.5. *If* $\varphi \in \mathrm{IBr}(B)$ *then* φ *is a* $\mathbb{Z}$*-linear combination of* $\{\chi^0 \mid \chi \in \mathrm{Irr}(B)\}$*.*

PROOF. See [**Nav98**, Corollary 2.16 and Lemma 3.16]. □

If $\chi \in \mathrm{Irr}(G)$ it is well known $\chi(1)$ divides $|G|$, however this is not true for Brauer characters (it is not even true that $\varphi(1)_p$ divides $|G|_p$).

PROBLEM 3.6. *Prove that* $p^{a-\mathrm{d}(B)} = \min\{\varphi(1)_p \mid \varphi \in \mathrm{IBr}(B)\}$*.*

Now, a brief digression to introduce the so-called p-adic valuation ν. If $n \in \mathbb{Z}\backslash\{0\}$ with $|n|_p = p^a$ then we write $\nu(n) = a$. Notice that $\nu(nm) = \nu(n) + \nu(m)$. We extend naturally $\nu : \mathbb{Q}^\times \to \mathbb{Z}$ by $\nu(p/q) = \nu(p) - \nu(q)$ and this still satisfies that $\nu(xy) = \nu(x) + \nu(y)$ for $x, y \in \mathbb{Q}^\times$. We take the convention that $\nu(0) = \infty$. With the p-adic valuation one can define an absolute value which in turn leads to the definition of the p-adic completion of $\mathbb{Q}$. Notice that if $\chi \in \mathrm{Irr}(B)$ then $\nu(\chi(1)) = \nu(|G|) - \mathrm{d}(B) + h_\chi$.

We have to define yet another set, in this case a maximal ideal of $\mathcal{S}$. Indeed, we write

$$\mathcal{P} := \{r/s \mid r \in M, s \in \mathbf{R}\backslash M\} = \ker({}^*\colon \mathcal{S} \to F).$$

The following is our necessary result on valuations.

Lemma 3.7. *The following hold.*

(i) $\mathcal{S} \cap \mathbb{Q} = \{q \in \mathbb{Q} \mid \nu(q) \geqslant 0\}$,

(ii) $\mathcal{P} \cap \mathbb{Q} = \{q \in \mathbb{Q} \mid \nu(q) > 0\}$,

(iii) $\{q \in \mathbb{Q} \mid \nu(q) = 0\}$ *is the set of units of the ring* $\mathcal{S} \cap \mathbb{Q}$.

Proof. See [**Nav98**, Lemma 3.21]. □

Before the proof of our current goal, we come up with a way to compute $a_B(\mathfrak{Cl}_G(x)^+)$ which will be essential. We have defined $e_B = (f_B)^*$ where $f_B = \sum_{\chi\in\mathrm{Irr}(B)} e_\chi$. Recall that by setting $\mathcal{A} = B$ we obtained in Section 2.1.2 a the decomposition $f_B = \sum_{x\in G/\sim} f_B(\mathfrak{Cl}_G(x)^+)\mathfrak{Cl}_G(x)^+$. Since $f_B(\mathfrak{Cl}_G(x)^+) \in \mathcal{S}$ by Proposition 2.5, we have that $a_B(\mathfrak{Cl}_G(x)^+) = f_B(\mathfrak{Cl}_G(x)^+)^*$. By the same result, if $x \in G\backslash G^0$ we obtain that $a_B(\mathfrak{Cl}_G(x)^+) = 0$ (this shows that defect classes must be formed by p-regular elements).

Therefore we focus on obtaining said formula for p-regular elements. By using Proposition 2.3 and the formula before Proposition 2.5, if $x \in G^0$, we obtain

$$a_B(\mathfrak{Cl}_G(x)^+) = \left(\frac{1}{|G|}\sum_{\varphi\in\mathrm{IBr}(B)} \varphi(x^{-1})\Phi_\varphi(1)\right)^*.$$

Notice that $\frac{\Phi_\varphi(1)}{|G|} \in \mathcal{S}$ because $|G|_p$ divides $\Phi_\varphi(1)$ by Corollary 1.13. Since $\varphi(x^{-1}) \in \mathbf{R}$ and * is a ring homomorphism, we obtain

$$a_B(\mathfrak{Cl}_G(x)^+) = \sum_{\varphi\in\mathrm{IBr}(B)} \left(\frac{\Phi_\varphi(1)}{|G|}\right)^* \varphi(x^{-1})^*. \tag{3.2.1}$$

Proof of Theorem 3.4 Let $a = \nu(|G|)$ and write $|D| = p^f$. Our goal is to show $f = \mathrm{d}(B)$. Let $x \in G$ be such that $\mathfrak{Cl}_G(x)$ be a defect class for B. We have that $D \in \delta(\mathfrak{Cl}_G(x))$, so $D^g \in \mathrm{Syl}_p(\mathbf{C}_G(x))$ for some $g \in G$. Since $|G : \mathbf{C}_G(x)| = |\mathfrak{Cl}_G(x)|$ we obtain

$$p^{a-f} = |\mathfrak{Cl}_G(x)|_p$$

or in other words, $a - f = \nu(|\mathfrak{Cl}_G(x)|)$.

Using that $\mathfrak{Cl}_G(x)$ is a defect class for B we have

$$\lambda_B(\mathfrak{Cl}_G(x)^+) \neq 0$$

so if $\chi \in \mathrm{Irr}(B)$ we have

$$\left(\frac{\chi(x)|\mathfrak{Cl}_G(x)|}{\chi(1)}\right)^* \neq 0.$$

which means that

$$\frac{\chi(x)|\mathfrak{Cl}_G(x)|}{\chi(1)} \in \mathcal{S}\backslash\mathcal{P}.$$

Recall that $\chi(x) \in \mathbf{R}$, so if

$$\frac{|\mathfrak{Cl}_G(x)|}{\chi(1)} \in \mathcal{P}$$

then, using that $\mathcal{P}$ is an ideal of $\mathcal{S}$, we have

$$\frac{\chi(x)|\mathfrak{Cl}_G(x)|}{\chi(1)} \in \mathcal{P}$$

which is false. This forces $\frac{|\mathfrak{Cl}_G(x)|}{\chi(1)} \notin \mathcal{P}$ so, by Lemma 3.7, $\nu\left(\frac{|\mathfrak{Cl}_G(x)|}{\chi(1)}\right) \leqslant 0$. Therefore $\nu(|\mathfrak{Cl}_G(x)|) = a - f \leqslant \nu(\chi(1)) = a - \mathrm{d}(B) + h_\chi$. By taking some χ of height zero we obtain that $\mathrm{d}(B) \leqslant f$.

Now since $\mathfrak{Cl}_G(x)$ is a defect class for B, then we also have $a_B(\mathfrak{Cl}_G(x)^+) \neq 0$. By using the expression from 3.2.1, we obtain that

$$\sum_{\varphi \in \mathrm{IBr}(B)} \left(\frac{\Phi_\varphi(1)}{|G|}\right)^* \varphi(x^{-1})^* \neq 0$$

so there is some $\varphi \in \mathrm{IBr}(B)$ with $\varphi(x^{-1}) \notin \mathcal{P}$ (because * vanishes in $\mathcal{P}$). Since φ is a linear combination of $\{\chi^0 \mid \chi \in \mathrm{Irr}(B)\}$ it follows that some $\chi \in \mathrm{Irr}(B)$ satisfies $\chi(x^{-1}) \notin \mathcal{P}$. Since $|\mathfrak{Cl}_G(x)| = |\mathfrak{Cl}_G(x^{-1})|$ we have

$$\frac{\chi(x^{-1})|\mathfrak{Cl}_G(x^{-1})|}{\chi(1)} = \frac{\overline{\chi(x)}|\mathfrak{Cl}_G(x^{-1})|}{\chi(1)} = \overline{\omega_\chi(\mathfrak{Cl}_G(x)^+)} \in \mathbf{R}$$

and $\mathcal{P}$ is an ideal of $\mathcal{S}$, it follows that $\frac{\chi(1)}{|\mathfrak{Cl}_G(x)|} \notin \mathcal{P}$, so we conclude that

$$\nu(\chi(1)) - \nu(|\mathfrak{Cl}_G(x)|) = \nu\left(\frac{\chi(1)}{|\mathfrak{Cl}_G(x)|}\right) \leqslant 0$$

by Lemma 3.7. Thus $\nu(\chi(1)) = a - \mathrm{d}(B) + h_\chi \leqslant a - f = \nu(|\mathfrak{Cl}_G(x)|)$ so $f \leqslant \mathrm{d}(B) - h_\chi$ and then $f \leqslant \mathrm{d}(B)$, as desired. □

3.3. $\mathbf{O}_p(G)$ is back

Recall that an element x of some F-algebra A is called nilpotent if there is some $n \in \mathbb{N}$ such that $x^n = 0$. Further, by [**Nav98**, Theorem 1.8], the Jacobson radical $\mathbf{J}(A)$ is the unique maximal nilpotent right ideal of A.

LEMMA 3.8. *Let $x \in G$ and assume that $\mathfrak{Cl}_G(x) \cap \mathbf{C}_G(\mathbf{O}_p(G)) = \varnothing$. Then $\mathfrak{Cl}_G(x)^+ \in \mathbf{J}(\mathbf{Z}(FG))$. In particular, $\mathfrak{Cl}_G(x)^+$ is nilpotent.*

PROOF. It suffices to check that $\mathfrak{Cl}_G(x)^+ \in \mathbf{J}(\mathbf{Z}(FG))$. By Theorem 2.11, we should check that $\mathfrak{Cl}_G(x)^+$ lies in the kernel of λ_B for all $B \in \mathrm{Bl}(G)$ (or equivalently, in the kernel of λ_φ for any $\varphi \in \mathrm{IBr}(G)$. Let $\mathcal{X}$ be an irreducible F-representation of G afforing φ. By the definition of λ_φ, we want to prove that $\mathcal{X}(\mathfrak{Cl}_G(x)^+) = 0$.

Write $P = \mathbf{O}_p(G)$. We have that P acts by conjugation on $\mathfrak{Cl}_G(x)$. Let $\Omega = \{x^y \mid y \in P\} \subseteq \mathfrak{Cl}_G(x)$ (that is, the P-orbit of x under this action). If $y \in P$ then $x^y = xx^{-1}y^{-1}xy \in xP$, that is, every P-conjugate of x is contained in xP. Now, we know that $P \subseteq \ker(\mathcal{X})$, so $\mathcal{X}(xt) = \mathcal{X}(x)$ for all $t \in P$, so $\mathcal{X}$ is constant on xP, and by the previous argument, it is constant on Ω. Since $|\Omega|$ is divisible by p (because $\mathfrak{Cl}_G(x) \cap \mathbf{C}_G(\mathbf{O}_p(G)) = \varnothing$), it follows that

$$\sum_{x_0 \in \Omega} \mathcal{X}(x_0) = |\Omega|\mathcal{X}(x) = 0$$

(because F has characteristic p). If $\Omega_1, \ldots, \Omega_t$ is the set of P-orbits on $\mathfrak{Cl}_G(x)$ with representatives $x_1, \ldots, x_t$, then by applying the previous argument to the x_i's we get

$$\mathcal{X}(\mathfrak{Cl}_G(x)^+) = \sum_{i=1}^{t} \left(\sum_{z \in \Omega_i} \mathcal{X}(z) \right) = \sum_{i=1}^{t} |\Omega_i|\mathcal{X}(x_i) = 0,$$

as desired. ☐

COROLLARY 3.9. *Let $B \in \mathrm{Bl}(G)$. Then $\mathbf{O}_p(G)$ is contained in every defect group of B.*

PROOF. Let $x \in G$ be such that $\mathfrak{Cl}_G(x)$ is a defect class for B. Since $\lambda_B(\mathfrak{Cl}_G(x)^+)$ by Lemma 3.8 we have that $\mathfrak{Cl}_G(x) \cap \mathbf{C}_G(\mathbf{O}_p(G)) \neq \varnothing$. Now $\mathbf{O}_p(G) \lhd G$ so $\mathbf{C}_G(\mathbf{O}_p(G)) \lhd G$ which means that $\mathfrak{Cl}_G(x) \subseteq \mathbf{C}_G(\mathbf{O}_p(G))$, so $\mathbf{O}_p(G) \subseteq \mathbf{C}_G(x)$. Thus if we take $D \in \mathrm{Syl}_p(\mathbf{C}_G(x))$ or any G-conjugate, we obtain $\mathbf{O}_p(G) \subseteq D$. ☐

We will not prove this, but there is a theorem of J. A. Green that states that if D is a defect group of some block of G contained in a Sylow p-subgroup P of G, then there is $x \in G^0$ such that $D = P^x \cap P$.

LECTURE 4

Brauer's first main theorem

Our new goal is proving the existence of a canonical bijection

$$\mathrm{Bl}(G|D) \to \mathrm{Bl}(\mathbf{N}_G(D)|D)$$

for a given defect group D. This is known as Brauer's first main theorem.

4.1. The Brauer map

Let P be a p-subgroup of G, and let $\mathbf{C}_G(P) \subseteq H \subseteq \mathbf{N}_G(P)$. The Brauer map is defined by

$$\begin{aligned} \mathrm{Br}_P : \mathbf{Z}(FG) &\to \mathbf{Z}(FH) \\ \mathfrak{Cl}_G(x)^+ &\mapsto \sum_{y \in \mathfrak{Cl}_G(x) \cap \mathbf{C}_G(P)} y \end{aligned}$$

where we set $\mathrm{Br}_P(\mathfrak{Cl}_G(x)^+) = 0$ if $\mathfrak{Cl}_G(x) \cap \mathbf{C}_G(P) = \varnothing$.

THEOREM 4.1. *The Brauer map is an algebra homomorphism.*

SKETCH OF PROOF. The map Br_P is clearly F-linear so we only need to show that for $x, y \in G/\sim$ we have

$$\mathrm{Br}_P(\mathfrak{Cl}_G(x)^+)\mathrm{Br}_P(\mathfrak{Cl}_G(y)^+) = \mathrm{Br}_P(\mathfrak{Cl}_G(x)^+\mathfrak{Cl}_G(y)^+).$$

For this, we write

$$\mathfrak{Cl}_G(x)^+\mathfrak{Cl}_G(y)^+ = \sum_{z \in G/\sim} a_{xyz}\mathfrak{Cl}_G(z)^+$$

and it turns out that the coefficient

$$a_{xyz} = |\{(x_0, y_0) \in \mathfrak{Cl}_G(x) \times \mathfrak{Cl}_G(y) \mid x_0y_0 = z\}|^*.$$

Write $C = \mathbf{C}_G(P)$. Now on one hand if $c \in C$ then the coefficient of c in

$$\mathrm{Br}_P(\mathfrak{Cl}_G(x)^+\mathfrak{Cl}_G(y)^+) = \sum_{z \in G/\sim} a_{xyz}\mathrm{Br}_P(\mathfrak{Cl}_G(z)^+)$$

is a_{xyc} (choosing the appropiate $G/\sim$ such that $c \in G/\sim$). On the other hand, the coefficient of c in $\mathrm{Br}_P(\mathfrak{Cl}_G(x)^+)\mathrm{Br}_P(\mathfrak{Cl}_G(y)^+)$ is

$$b_{xyc} = |\{(x_0, y_0) \in (\mathfrak{Cl}_G(x) \cap C) \times (\mathfrak{Cl}_G(y) \cap C) \mid x_0y_0 = c\}|^*.$$

The final trick is that P acts by conjugation on

$$\{(x_0, y_0) \in \mathfrak{Cl}_G(x) \times \mathfrak{Cl}_G(y) \mid x_0 y_0 = z\}$$

with fixed points

$$\{(x_0, y_0) \in (\mathfrak{Cl}_G(x) \cap C) \times (\mathfrak{Cl}_G(y) \cap C) \mid x_0 y_0 = c\}$$

and since P is a p-group, this implies that these two sets have sizes congruent modulo p, so $a_{xyz} = b_{xyz}$. □

Remarkably, the previous proof heavily relies on our field having characteristic p.

LEMMA 4.2. *Let $P \leqslant G$ be a p-subgroup. For $x \in G/\sim$ choose some $D_x \in \delta(\mathfrak{Cl}_G(x))$. Then*

$$\ker(\mathrm{Br}_P) = \sum_{P \nsubseteq_G D_x} F\mathfrak{Cl}_G(x)^+ (= \langle \mathfrak{Cl}_G(x)^+ \mid P \nsubseteq_G D_x \rangle_F)$$

(the RHS denotes the linear combinations of the class sums $\mathfrak{Cl}_G(x)^+$ with $P \nsubseteq_G D_x$).

PROOF. We claim that $\mathrm{Br}_P(\mathfrak{Cl}_G(x)^+) \neq 0$ if and only if $P \subseteq_G D_x$

Indeed, $\mathrm{Br}_P(\mathfrak{Cl}_G(x)^+) \neq 0$ if and only if $\mathfrak{Cl}_G(x) \cap \mathbf{C}_G(P) \neq \varnothing$, which happens if and only if some G-conjugate x^t of x centralizes P. This happens if and only if $P \subseteq \mathbf{C}_G(x^t)$ and then $P \subseteq D_0 \in \mathrm{Syl}_p(\mathbf{C}_G(x^t))$ and this happens if and only if $P \subseteq_G D_x$.

Now if $z \in \mathbf{Z}(FG)$ satisfies $\mathrm{Br}_P(z)$ and we write

$$z = \sum_{x \in G/\sim} z_x \mathfrak{Cl}_G(x)^+$$

then notice that $z_x = 0$ if $\mathrm{Br}_P(\mathfrak{Cl}_G(x)^+) \neq 0$ because the sets $\mathfrak{Cl}_G(x) \cap C$ are disjoint. By the first paragraph, if $\mathrm{Br}_P(z) = 0$ then $z_x = 0$ whenever $P \subseteq_G D_x$. Conversely, any linear combination

$$z = \sum_{P \nsubseteq D_x} z_x \mathfrak{Cl}_G(x)^+$$

satisfies $\mathrm{Br}_P(z) = \sum_{P \nsubseteq D_x} z_x \mathrm{Br}_P(\mathfrak{Cl}_G(x)^+) = 0$. □

THEOREM 4.3. *Let $B \in \mathrm{Bl}(G|D)$ and let $P \leqslant G$ be a p-subgroup. Then $\mathrm{Br}_P(e_B) \neq 0$ if and only if $P \subseteq_G D$.*

PROOF. Write (as usual)

$$e_B = \sum_{x \in G/\sim} a_B(\mathfrak{Cl}_G(x)^+) \mathfrak{Cl}_G(x)^+.$$

By Lemma 4.2, $\mathrm{Br}_P(e_B) \neq 0$ if and only if there is some $x \in \mathfrak{Cl}_G(x)^+$ with $P \subseteq_G D_x \in \delta(\mathfrak{Cl}_G(x))$ and with $a_B(\mathfrak{Cl}_G(x)^+) \neq 0$.

First suppose that $P \subseteq_G D$. By choosing a defect class $\mathfrak{Cl}_G(x)$ of B, we have $a_B(\mathfrak{Cl}_G(x)^+) \neq 0$ and $P \subseteq_G D \in \delta(\mathfrak{Cl}_G(x))$. By the previous paragraph this implies $\mathrm{Br}_P(e_B) \neq 0$.

Conversely, if there is some class $\mathfrak{Cl}_G(x) \neq 0$, $P \subseteq_G D_x \in \delta(\mathfrak{Cl}_G(x))$ and with $a_B(\mathfrak{Cl}_G(x)^+) \neq 0$ then by the Min–Max theorem we have $D_x \subseteq_G D$ and therefore $P \subseteq_G D_x \subseteq_G D$ and we are done. □

We have found another definition of the defect groups: the maximal p-subgroup D of G up to G-conjugation with $\mathrm{Br}_D(e_B) \neq 0$.

4.2. Block induction

Let $H \leqslant G$ and $b \in \mathrm{Bl}(H)$. We use the algebra homomorphism $\lambda_b : \mathbf{Z}(FH) \to F$ to define an F-linear map

$$\lambda_b^G : \mathbf{Z}(FG) \to F$$
$$\mathfrak{Cl}_G(x)^+ \mapsto \lambda_b\left(\sum_{y \in \mathfrak{Cl}_G(x) \cap H} y\right).$$

It may happen that λ_b^G is in fact an algebra homomorphism. In that case, by Theorem 2.11 we know that there is a unique $b^G \in \mathrm{Bl}(G)$ such that $\lambda_{b^G} = \lambda_b^G$. We say in this case that the **induced block** b^G is **defined**.

There is more than one definition of induced block in the literature, but they coincide in the important cases.

LEMMA 4.4. *Let $b \in \mathrm{Bl}(H)$ with $H \leqslant G$. If b^G is defined then every defect group of b is contained in a defect group of b^G.*

PROOF. Let $\mathfrak{Cl}_G(x)$ be a defect class for b^G, which implies that $\lambda_{b^G}(\mathfrak{Cl}_G(x)^+) \neq 0$. In particular

$$\lambda_b^G\left(\sum_{y \in \mathfrak{Cl}_G(x) \cap H} y\right) \neq 0.$$

Therefore there is some $\mathfrak{Cl}_H(z) \subseteq \mathfrak{Cl}_G(x) \cap H$ with $\lambda_b(\mathfrak{Cl}_H(z)) \neq 0$. By the Min–Max theorem we have $D_b \subseteq_H D_z \in \delta(\mathfrak{Cl}_H(z))$ where $D_b \in \delta(b)$. Now $D_z \subseteq \mathbf{C}_H(z) \subseteq \mathbf{C}_G(z)$, so there is $D \in \mathrm{Syl}_p(\mathbf{C}_G(z))$ with $D_z \subseteq D$. Now since $\mathfrak{Cl}_G(x) = \mathfrak{Cl}_G(z)$ is a defect class of B, then $\delta(B)$ contains D and we have $D_b \subseteq_G D$, as desired. □

Next is the big theorem.

Theorem 4.5. *Let $P \leqslant G$ be a p-subgroup, and let $H \leqslant G$ be such that $P\mathbf{C}_G(P) \subseteq H \subseteq \mathbf{N}_G(P)$. If $b \in \mathrm{Bl}(H)$ then b^G is defined and $\lambda_b^G = \lambda_b \circ \mathrm{Br}_P$. Further, if $B \in \mathrm{Bl}(G)$ then $B = b^G$ for some block $b \in \mathrm{Bl}(H)$ if and only if $P \subseteq D \in \delta(B)$.*

Proof. Since Br_P and λ_b are algebra homomorphisms, to prove that b^G is defined (that is, λ_b^G is an algebra homomorphism) it suffices to show that $\lambda_b^G = \lambda_b \circ \mathrm{Br}_P$.

Let $C = \mathbf{C}_G(P)$. If $x \in G$ then we wish to prove

$$\lambda_b\left(\sum_{y\in\mathfrak{Cl}_G(x)\cap H} y\right) = \lambda_b\left(\sum_{y\in\mathfrak{Cl}_G(x)\cap C} y\right).$$

Since $C \lhd H$ then $\mathfrak{Cl}_G(x) \cap C$ is a union of H-conjugacy classes, so we can split the LHS as follows

$$\lambda_b\left(\sum_{y\in\mathfrak{Cl}_G(x)\cap C} y\right) + \lambda_b\left(\sum_{y\in(\mathfrak{Cl}_G(x)\cap H)\setminus(\mathfrak{Cl}_G(x)\cap C)} y\right)$$

and it suffices to show that the rightmost term vanishes. Now let $z \in (\mathfrak{Cl}_G(x) \cap H)\setminus(\mathfrak{Cl}_G(x)\cap C)$. We have that $\mathfrak{Cl}_H(z)\cap C = \varnothing$. Now $P \lhd H$ so $P \subseteq \mathbf{O}_p(H)$, and therefore $\mathbf{C}_H(\mathbf{O}_p(H)) \subseteq \mathbf{C}_H(P) \subseteq C$. This implies that $\mathbf{C}_H(\mathbf{O}_p(H)) \cap \mathfrak{Cl}_H(z) = \varnothing$. By Lemma 3.8, $\mathfrak{Cl}_H(z)^+$ lies in the kernel of of λ_b, so $\lambda_b(\mathfrak{Cl}_H(z)^+) = 0$. The desired equality now follows since the rightmost term of the above sum vanishes, and the first part is proved.

The second part is a double implication so we start with the easy one. If $b^G = B$ for some $b \in \mathrm{Bl}(H)$, then we now from Lemma 4.4 that if $D_b \in \delta(b)$ then $D_b \subseteq D \in \delta(B)$. Now $P \lhd H$ so $P \subseteq \mathbf{O}_p(H) \subseteq D_b$ and it follows that $P \subseteq D$, as desired. Conversely, if $P \subseteq D \in \delta(B)$ then from Theorem 4.3 we have that $\mathrm{Br}_P(e_B) \neq 0$. Since Br_P is an algebra homomorphism, $\mathrm{Br}(e_B) \in \mathbf{Z}(FH)$ must be an idempotent, and therefore it is a sum of (different) primitive idempotents, so $\mathrm{Br}(e_B) = e_{b_1} + \cdots + e_{b_t}$ for some blocks $b_i \in \mathrm{Bl}(H)$, by Theorem 2.11. From the first part, if b is one of the b_i's, b^G is defined and $\lambda_b^G(e_B) = \lambda_b(\mathrm{Br}_P(e_B)) = \sum \lambda_b(e_{b_i}) = 1$ so $B = b^G$ by Theorem 2.11. □

From the above argument we get the following conclusion.

Corollary 4.6. *Let G, P, H be as before. Then*

$$\mathrm{Br}_P(e_B) = \sum_{b^G=B} e_b.$$

The following is treated by some experts as folklore, but it is a (nontrivial) application of the previous result.

Problem 4.7. *Let G be a finite group. If there is some p-subgroup $P \lhd G$ with $\mathbf{C}_G(P) \subseteq P$ then G has a unique block.*

As a consequence, if G is p-solvable and $\mathbf{O}_{p'}(G) = 1$ then G has a unique block (since Hall–Higman's Lemma 1.2.3 implies that $\mathbf{C}_G(\mathbf{O}_p(G)) \subseteq \mathbf{O}_p(G)$).

Problem 4.8. *Let $P \lhd G$ be a p-subgroup and $B \in \mathrm{Bl}(G|P)$. Then*

$$e_B = \sum_{P \in \delta(\mathfrak{Cl}_G(x))} a_B(\mathfrak{Cl}_G(x)^+)\mathfrak{Cl}_G(x)^+.$$

4.3. The first main theorem

The following is a key result in group theory, for a proof see [**Nav98**, Theorem 4.16]. In a sense it is a version of the first main theorem but for conjugacy classes. We denote by $\mathrm{Cl}(G|D)$ the set of conjugay classes of G with defect group D.

Theorem 4.9. *Let $D \leqslant G$ be a p-subgroup. The map $\mathfrak{Cl}_G(x) \mapsto \mathfrak{Cl}_G(x) \cap \mathbf{C}_G(D)$ is a bijection $\mathrm{Cl}(G|D) \to \mathrm{Cl}(\mathbf{N}_G(D)|D)$.*

Notice that if $\mathfrak{Cl}_G(x)$ is a conjugacy class with defect group D, then

$$\mathrm{Br}_D(\mathfrak{Cl}_G(x)^+) = (\mathfrak{Cl}_G(x) \cap \mathbf{C}_G(D))^+.$$

We are finally ready to prove Brauer's first main theorem. Recall that $\mathrm{Bl}(G|D)$ denotes the (possibly empty) set of blocks of G with defect group D.

Theorem 4.10 (Brauer's first main). *The map*

$$\mathrm{Bl}(\mathbf{N}_G(D)|D) \to \mathrm{Bl}(G|D)$$
$$b \mapsto b^G$$

is a bijection. Its inverse is given by applying Br_D to the block idempotents.

Proof. Denote by $N = \mathbf{N}_G(D)$ and $C = \mathbf{C}_G(D)$. By Theorem 4.5 we know that if $b \in \mathrm{Bl}(N|D)$ then b^G is defined and $\lambda_{b^G} = \lambda_b^G = \lambda_b \circ \mathrm{Br}_D$.

Claim 1: b^G has defect group D (so the map is well defined).

Let $\mathfrak{Cl}_N(y)$ be a defect class for b. Since b has defect group D then so does $\mathfrak{Cl}_N(y)$ and therefore $\mathfrak{Cl}_G(y)$ has defect group D and $\mathfrak{Cl}_N(y) = \mathfrak{Cl}_G(y) \cap C$ by Theorem 4.9. Now

$$\lambda_{b^G}(\mathfrak{Cl}_G(y)^+) = \lambda_b(\mathrm{Br}_D(\mathfrak{Cl}_G(y)^+)) = \lambda_b((\mathfrak{Cl}_G(y) \cap C)^+) = \lambda_b(\mathfrak{Cl}_N(y)^+) \neq 0$$

so by the Min–Max theorem we have that a defect group of b^G is contained in D. By Lemma 4.4, D is contained in some defect group of b^G and this proves the claim.

Claim 2: The map $b \mapsto b^G$ is surjective.

Let $B \in \mathrm{Bl}(G|D)$. By Theorem 4.5 there is some $b \in \mathrm{Bl}(N)$ with $b^G = B$. We need to prove that b has defect group D. Now since $D \lhd N$ then D is contained in the defect groups of b. By Lemma 4.4, the defect groups of b are contained in the defect groups of b. This shows that b has defect group D, as desired.

Claim 3: The map $b \mapsto b^G$ is injective.

Let $b, c \in \mathrm{Bl}(N|D)$ and assume $b^G = c^G$. This implies that $\lambda_{b^G} = \lambda_b \circ \mathrm{Br}_D = \lambda_c \circ \mathrm{Br}_D = \lambda_{c^G}$ by Theorem 4.5. If $\mathfrak{Cl}_G(x)$ has defect group D then

$$\lambda_b((\mathfrak{Cl}_G(x) \cap C)^+) = \lambda_c((\mathfrak{Cl}_G(x) \cap C)^+)$$

so by Theorem 4.9, λ_b and λ_c coincide in every conjugacy class of N with defect group D. Now using Problem 4.8

$$1 = \lambda_b(e_b) = \sum_{D \in \delta(\mathfrak{Cl}_N(y))} a_b(\mathfrak{Cl}_N(y)^+)\lambda_b(\mathfrak{Cl}_N(y)^+) =$$

$$= \sum_{D \in \delta(\mathfrak{Cl}_N(y))} a_b(\mathfrak{Cl}_N(y)^+)\lambda_c(\mathfrak{Cl}_N(y)^+) = \lambda_c(e_b)$$

which shows $e_b = e_c$ so $b = c$. □

As a consequence of the first main theorem, defect groups are p-radical (we say a p-subgroup P is p-radical if $P = \mathbf{O}_p(\mathbf{N}_G(P))$).

PROBLEM 4.11. *Let $D \in \delta(B)$ for some block B. Prove that $\mathbf{O}_p(\mathbf{N}_G(D)) = D$. (Hint: $\mathbf{O}_p(G)$ is contained in all the defect groups!).*

LECTURE 5

The principal block and Brauer's third main theorem

The principal block is the unique block $B_0(G)$ that contains the trivial character 1_G. Its defect groups are the Sylow p-subgroups. In some sense, it is the most important block of a group in terms of the structural information it contains (if time permits, we might devote a session at the end of the course to prove things about principal blocks).

Assume $H \leqslant G$ and $b \in \mathrm{Bl}(H)$ is such that b^G is defined and $b^G = B_0(G)$. Brauer's third main theorem shows that this forces $b = B_0(H)$.

Brauer's proof [**Bra64**] (which appeared in the first volume of the Journal of Algebra) relies on results about block coverings and assumes that $\mathbf{C}_G(P) \subseteq H$ for some p-subgroup $P \leqslant G$, so we follow the proof from [**Nav98**, Chapter 6], which is a bit more convoluted but also gives interesting insights on induced blocks. Since we are taking two weeks off after this session I believe this is the more natural approach (instead of introducing stuff about normal subgroups and coverings and then taking a break). I still think that it is worth reading Brauer's paper, which has aged beautifully.

5.1. Preliminary results

In this section, for any $\chi \in \mathrm{Char}(G)$, we write

$$\begin{aligned} \omega_\chi : \mathbf{Z}(\mathbb{C}G) &\to \mathbb{C} \\ \mathfrak{Cl}_G(x)^+ &\mapsto \frac{|\mathfrak{Cl}_G(x)|\chi(x)}{\chi(1)}. \end{aligned}$$

Notice that $\omega_\chi(\mathfrak{Cl}_G(x)^+)$ may not be in $\mathbf{R}$ if χ is not irreducible. Observe that

$$\chi(1)\omega_\chi(\mathfrak{Cl}_G(x)^+) = |\mathfrak{Cl}_G(x)|\chi(x) = \sum_{\psi \in \mathrm{Irr}(G)} |\mathfrak{Cl}_G(x)|[\chi,\psi]\psi(x) =$$
$$= \sum_{\psi \in \mathrm{Irr}(G)} [\chi,\psi]\psi(1)\omega_\psi(\mathfrak{Cl}_G(x)^+)$$

so $\chi(1)\omega_\chi = \sum_{\psi \in \mathrm{Irr}(G)}[\chi,\psi]\psi(1)\omega_\psi$. We shall use this fact throughout this section withour further mention.

If $H \leqslant G$ and any field K and any K-linear map $\lambda : \mathbf{Z}(KH) \to K$ we denote by $\lambda^G : \mathbf{Z}(KG) \to K$ the K-linear map defined by

$$\lambda^G(\mathfrak{Cl}_G(x)^+) = \lambda((\mathfrak{Cl}_G(x) \cap H)^+).$$

Notice that if $b \in \mathrm{Bl}(H)$ then the induced map λ_b^G is just $(\lambda_b)^G$.

Lemma 5.1. *Let $H \leqslant G$ and $\xi \in \mathrm{Irr}(H)$. Then $\omega_\xi^G = \omega_{\xi^G}$.*

Proof. We have

$$\xi^G(1)\omega_{\xi^G} = \sum_{\psi \in \mathrm{Irr}(G)} [\xi^G, \psi]\psi(1)\omega_\psi$$

so we need to show that the RHS equals $\xi^G(1)(\omega_\xi)^G$.

Let $x \in G$ and notice that if $\mathfrak{Cl}_G(x) \cap H$ is nonempty then we may write

$$\mathfrak{Cl}_G(x) \cap H = \bigcup_{i=1}^{t} \mathfrak{Cl}_H(x_i)$$

as a disjoint union. With this expression, we may rewrite the induction formula

$$\xi^G(x) = |\mathbf{C}_G(x)| \sum_{i=1}^{t} \frac{\xi(x_i)}{|\mathbf{C}_H(x_i)|}$$

(I recommend trying to prove this as a problem, but it can be seen in [**Isa06**, p. 64]). Thus

$$\sum_{\psi \in \mathrm{Irr}(G)} [\xi^G, \psi]\psi(1)\omega_\psi(\mathfrak{Cl}_G(x)^+) = \sum_{\psi \in \mathrm{Irr}(G)} [\xi^G, \psi]\psi(1)\frac{|\mathfrak{Cl}_G(x)|\psi(x)}{\psi(1)} =$$
$$= |\mathfrak{Cl}_G(x)|\xi^G(x) = \frac{|G|}{|\mathbf{C}_G(x)|}|\mathbf{C}_G(x)| \sum_{i=1}^{t} \frac{\xi(x_i)}{|\mathbf{C}_H(x_i)|} = |G| \sum_{i=1}^{t} \frac{\xi(x_i)}{|\mathbf{C}_H(x_i)|} =$$
$$= |G| \sum_{i=1}^{t} \frac{|\mathfrak{Cl}_H(x_i)|\xi(x_i)}{|H|} = \frac{|G|}{|H|} \sum_{i=1}^{t} |\mathfrak{Cl}_H(x_i)|\xi(x_i) = \xi^G(1) \sum_{i=1}^{t} \omega_\xi(\mathfrak{Cl}_H(x_i)^+) =$$
$$= \xi^G(1)\omega_\xi((\mathfrak{Cl}_G(x) \cap H)^+) = \xi^G(1)(\omega_\xi)^G(\mathfrak{Cl}_G(x)^+)$$

where we have used that $\xi^G(1) = |G : H|\xi(1)$. □

Corollary 5.2. *If $H \leqslant G$ and $\xi \in \mathrm{Irr}(H)$ then $\omega_{\xi^G}(\mathfrak{Cl}_G(x)^+) \in \mathbf{R}$. Therefore, if $z \in \mathbf{Z}(\mathcal{S}G)$, $\omega_{\xi^G}(z) \in \mathcal{S}$.*

Proof. By the previous result (and inheriting the notation in the proof)

$$\omega_{\xi^G}(\mathfrak{Cl}_G(x)^+) = \omega_\xi((\mathfrak{Cl}_G(x) \cap H)^+) = \sum_{i=1}^{t} \omega_\xi(\mathfrak{Cl}_H(x_i)^+)$$

which is an algebraic integer. The second part follows by linearity. □

Corollary 5.3. *Let $H \leqslant G$ and $\xi \in \mathrm{Irr}(H)$. If $\xi^G \in \mathrm{Irr}(G)$ then $\mathrm{bl}(\xi)^G$ is defined and contains ξ^G.*

Proof. Since ξ^G is irreducible

$$\omega_{\xi^G} : \mathbf{Z}(\mathbb{C}G) \to \mathbb{C}$$

is an algebra homomorphism. Further

$$\lambda_b^G(\mathfrak{Cl}_G(x)^+) = \lambda_b((\mathfrak{Cl}_G(x) \cap H)^+) = \omega_\xi((\mathfrak{Cl}_G(x) \cap H)^+)^* = \omega_\xi^G(\mathfrak{Cl}_G(x)^+)^* = \\ = \omega_{\xi^G}(\mathfrak{Cl}_G(x)^+)^* = \lambda_{\xi^G}(\mathfrak{Cl}_G(x)^+)$$

which implies λ_b^G is an algebra homomorphism $\mathbf{Z}(FG) \to F$. It follows that b^G is defined and since $\lambda_b^G = \lambda_{b^G} = \lambda_{\xi^G}$, then b^G contains ξ^G. □

If $B \in \mathrm{Bl}(G)$ and $\chi \in \mathrm{Char}(G)$ then we denote by

$$\chi_B = \sum_{\psi \in \mathrm{Irr}(B)} [\chi, \psi]\psi$$

so that $\chi = \sum_{B \in \mathrm{Bl}(G)} \chi_B$. Recall that $\mathcal{P} = \ker({}^*\colon \mathcal{S} \to F)$.

Lemma 5.4. *Let $H \leqslant G$ and consider $b \in \mathrm{Bl}(H)$, $\xi \in \mathrm{Irr}(b)$. Let $B \in \mathrm{Bl}(G)$.*

(i) *We have*

$$\frac{|\mathfrak{Cl}_G(x)|(\xi^G)_B(x)}{\xi^G(1)} \in \mathcal{S}.$$

(ii) *If b^G is defined then*

(a) if $b^G = B$ and $\chi \in \mathrm{Irr}(B)$ then

$$\frac{|\mathfrak{Cl}_G(x)|(\xi^G)_B(x)}{\xi^G(1)} \equiv \frac{|\mathfrak{Cl}_G(x)|\chi(x)}{\chi(1)} \bmod \mathcal{P}$$

(b) if $b^G \neq B$ then

$$\frac{|\mathfrak{Cl}_G(x)|(\xi^G)_B(x)}{\xi^G(1)} \in \mathcal{P}.$$

Proof. Recall that we wrote $f_B = \sum_{\chi \in \mathrm{Irr}(B)} e_\chi$. The key to this proof is that

$$\frac{|\mathfrak{Cl}_G(x)|(\xi^G)_B(x)}{\xi^G(1)} = \omega_\xi^G(f_B \mathfrak{Cl}_G(x)^+).$$

Indeed, using that ω_χ is an algebra homomorphism if χ is irreducible, and that $\omega_\chi(f_B) = \delta_{\mathrm{bl}(\chi),B}$ we obtain

$$\begin{aligned}\xi^G(1)\omega_{\xi^G}(f_B\mathfrak{Cl}_G(x)^+) &= \sum_{\chi\in\mathrm{Irr}(G)}[\xi^G,\chi]\chi(1)\omega_\chi(f_B\mathfrak{Cl}_G(x)^+) = \\ &= \sum_{\chi\in\mathrm{Irr}(B)}[\xi^G,\chi]\chi(1)\omega_\chi(\mathfrak{Cl}_G(x)^+) = \sum_{\chi\in\mathrm{Irr}(B)}[\xi^G,\chi]|\mathfrak{Cl}_G(x)|\chi(x) = \\ &= |\mathfrak{Cl}_G(x)|(\xi^G)_B(x)\end{aligned}$$

and the claim follows. Now f_B has coefficients in $\mathcal{S}$, and therefore so does $f_B\mathfrak{Cl}_G(x)^+$. By Corollary 5.2, $\omega_{\xi^G}(f_B\mathfrak{Cl}_G(x)^+) \in \mathcal{S}$, and (i) is proved.

Assume b^G is defined. We have

$$\begin{aligned}\lambda_b^G(\mathfrak{Cl}_G(x)^+) = \lambda_b((\mathfrak{Cl}_G(x)\cap H)^+) &= \lambda_\xi((\mathfrak{Cl}_G(x)\cap H)^+) = \\ &= \omega_\xi((\mathfrak{Cl}_G(x)\cap H)^+)^* = \omega_{\xi^G}(\mathfrak{Cl}_G(x)^+)^*\end{aligned}$$

so if $z \in \mathbf{Z}(\mathcal{S}G)$ then $\lambda_b^G(z^*) = \omega_{\xi^G}(z)^*$. By the proof of the first part we have

$$\lambda_{b^G}(e_B\mathfrak{Cl}_G(x)^+) = \lambda_{b^G}((f_B\mathfrak{Cl}_G(x)^+)^*) = \omega_{\xi^G}(f_B\mathfrak{Cl}_G(x)^+) = \left(\frac{|\mathfrak{Cl}_G(x)|(\xi^G)_B(x)}{\xi^G(1)}\right)^*.$$

If $b^G = B$ then $\lambda_{b^G}(e_B) = 1$, so if we take $\chi \in \mathrm{Irr}(B)$ we have

$$\left(\frac{|\mathfrak{Cl}_G(x)|\chi(x)}{\chi(1)}\right)^* = \lambda_b(\mathfrak{Cl}_G(x)^+) = \lambda_b(e_B\mathfrak{Cl}_G(x)^+) = \left(\frac{|\mathfrak{Cl}_G(x)|(\xi^G)_B(x)}{\xi^G(1)}\right)^*.$$

and (ii)(a) follows.

If $b^G \neq B$ then $\lambda_{b^G}(e_B) = 0$ and therefore

$$0 = \lambda_b^G(e_B)\lambda_b^G(\mathfrak{Cl}_G(x)^+) = \left(\frac{|\mathfrak{Cl}_G(x)|(\xi^G)_B(x)}{\xi^G(1)}\right)^*$$

and (ii)(b) follows. □

We obtain an interesting and very useful consequence.

PROBLEM 5.5. *Assume* $H \leqslant G$, $b \in \mathrm{Bl}(H)$ *and suppose* b^G *is defined. Let* $\xi \in \mathrm{Irr}(b)$ *and* $B \in \mathrm{Bl}(G)$. *Prove that*

(i) $\nu((\xi^G)_B(1)) > \nu(\xi^G(1))$ *if* $B \neq b^G$,

(ii) $\nu((\xi^G)_B(1)) = \nu(\xi^G(1))$ *if* $B = b^G$.

Deduce that there is some $\chi \in \mathrm{Irr}(b^G)$ *such that* $[\chi, \xi^G] \neq 0$.

(Hint: Apply the previous result to $x = 1$ and use the result on valuations.)

5.2. A generalized character

If $\chi \in \mathrm{Char}(G)$ we define $\widehat{\chi}(x) = |G|_p\chi(x)$ if $g \in G^0$ and $\widehat{\chi}(x) = 0$ otherwise. We will not prove anything here, but state the main result that we need.

PROPOSITION 5.6. *Let $\chi \in \mathrm{Irr}(B)$ for $B \in \mathrm{Bl}(G)$. Then*

(i) *$\widehat{\chi}$ is a $\mathbb{Z}$-linear combination of $\mathrm{Irr}(B)$,*

(ii) *χ has height zero if and only if $\nu([\widehat{\chi}, \xi]) = \nu(\xi(1))$ for all $\xi \in \mathrm{Irr}(B)$ (equivalently, $\frac{[\widehat{\chi},\xi]}{\xi(1)} \in \mathcal{S}\backslash\mathcal{P}$).*

PROOF. See [**Nav98**, Lemma 3.20 and Theorem 3.24]. □

We can use this generalized character to test whether a (height zero) character belongs to an induced block.

PROPOSITION 5.7. *Let $H \leqslant G$ and assume b^G is defined for some $b \in \mathrm{Bl}(H)$. Let $\chi \in \mathrm{Irr}(G)$ and $\psi \in \mathrm{Irr}(b)$.*

(i) *If $\chi \notin \mathrm{Irr}(b^G)$ then*

$$\frac{[\widehat{\chi_H}, \psi]}{\psi(1)} \in \mathcal{P},$$

(ii) *if $\chi \in \mathrm{Irr}(b^G)$ has height zero then*

$$\frac{[\widehat{\chi_H}, \psi]}{\psi(1)} \not\equiv 0 \bmod \mathcal{P}.$$

PROOF. First notice that if $x \in H^0$ then

$$\widehat{\chi_H}(x) = \widehat{\chi}(x) = |G|_p\chi(x)$$

and

$$\widehat{\chi}_H(x) = |H|_p\chi(x)$$

and if both $\widehat{\chi}_H$ and $\widehat{\chi_H}$ vanish in $H\backslash H^0$. This shows that

$$\widehat{\chi_H} = \frac{1}{|G:H|_p}\widehat{\chi}_H.$$

Secondly, recall that by Proposition 5.6, $\widehat{\chi}$ is a $\mathbb{Z}$-linear combination of $\mathrm{Irr}(B)$ and, in particular, for any $\eta \in \mathrm{Char}(G)$ we have

$$[\widehat{\chi}, \eta] = [\widehat{\chi}, \eta_B].$$

We have

$$\frac{[\widehat{\chi_H}, \psi]}{\psi(1)} = \frac{[\widehat{\chi}_H, \psi]}{|G:H|_p\psi(1)} = \frac{[\widehat{\chi}, \psi^G]}{|G:H|_p\psi(1)} = \frac{[\widehat{\chi}, (\psi^G)_B]}{|G:H|_p\psi(1)}$$

and using that $\psi^G(1) = |G:H|\psi(1)$ we can write

$$\frac{[\widehat{\chi}, (\psi^G)_B]}{|G:H|_p \psi(1)} = \frac{|G:H|[\widehat{\chi}, (\psi^G)_B]}{|G:H|_p \psi^G(1)} = \frac{|G:H|_{p'}[\widehat{\chi}, (\psi^G)_B]}{\psi^G(1)} =$$
$$= \frac{|G:H|_{p'}}{\psi^G(1)|G|} \sum_{x \in G/\sim} |\mathfrak{Cl}_G(x)|(\psi^G)_B(x)\widehat{\chi}(x^{-1}).$$

Now, $\widehat{\chi}$ vanishes in $G \backslash G^0$ and $\widehat{\chi}(g) = |G|_p \chi(g)$ otherwise. We can rewrite the last expression as

$$\frac{|G:H|_{p'}|G|_p}{\psi^G(1)|G|} \sum_{x \in G^0/\sim} |\mathfrak{Cl}_G(x)|(\psi^G)_B(x)\chi(x^{-1}) =$$
$$= \frac{1}{|H|_{p'}} \sum_{x \in G^0/\sim} \frac{|\mathfrak{Cl}_G(x)|(\psi^G)_B(x)}{\psi^G(1)} \chi(x^{-1}).$$

Now, if $B \neq b^G$ then

$$\frac{|\mathfrak{Cl}_G(x)|(\psi^G)_B(x)}{\psi^G(1)} \equiv 0 \bmod \mathcal{P}$$

by Lemma 5.4 and we conclude that $\frac{[\widehat{\chi_H}, \psi]}{\psi(1)} \equiv 0 \bmod \mathcal{P}$.

Otherwise, by the same result,

$$\frac{|\mathfrak{Cl}_G(x)|(\psi^G)_B(x)}{\psi^G(1)} \equiv \frac{|\mathfrak{Cl}_G(x)|\chi(x)}{\chi(1)} \bmod \mathcal{P}.$$

Now, since $\widehat{\chi}$ vanishes outside G^0,

$$\frac{[\widehat{\chi_H}, \psi]}{\psi(1)} \equiv \frac{1}{|H|_{p'}} \sum_{x \in G^0/\sim} \frac{|\mathfrak{Cl}_G(x)|\chi(x)}{\chi(1)} \chi(x^{-1}) =$$
$$= \frac{1}{|H|_{p'}\chi(1)|G|_p} \sum_{x \in G/\sim} |\mathfrak{Cl}_G(x)|\widehat{\chi}(x)\chi(x^{-1}) =$$
$$= \frac{|G|}{|H|_{p'}\chi(1)|G|_p}[\widehat{\chi}, \chi] = \frac{|G|_{p'}}{|H|_{p'}} \frac{[\widehat{\chi}, \chi]}{\chi(1)} \not\equiv 0 \bmod \mathcal{P}$$

by Proposition 5.6. □

5.3. The proof

We prove a more general result, due to Okuyama.

Theorem 5.8 (Okuyama). *Let $H \leqslant G$, $\chi \in \mathrm{Irr}(G)$ and assume $\chi_H \in \mathrm{Irr}(H)$. Assume further that $h_\chi = 0 = h_{\chi_H}$ in their respective blocks. If $c \in \mathrm{Bl}(H)$ is such that c^G is defined, then $c = \mathrm{bl}(\chi_H)$ if and only if $e^G = \mathrm{bl}(\chi)$.*

Proof. If $c = \mathrm{bl}(\chi_H)$ then $\mathrm{bl}(\chi_H)^G$ is defined, and we need to show $\mathrm{bl}(\chi_H)^G = \mathrm{bl}(\chi)$. Now χ_H has height zero, so by Proposition 5.6,

$$\frac{[\widehat{\chi_H}, \chi_H]}{\chi_H(1)} \neq 0 \bmod \mathcal{P}$$

and since χ has height zero, Proposition 5.7 implies $\chi \in \mathrm{bl}(\chi_H)^G$, as desired.

If $c^G = \mathrm{bl}(\chi)$ then we need $e = \mathrm{bl}(\chi_H)$. Let $\psi \in \mathrm{Irr}(c)$. Since $\chi \in c^G$ has height zero, we have

$$[\widehat{\chi_H}, \psi] \neq 0.$$

Now, by Proposition 5.6, $\widehat{\chi_H}$ only involves characters in $\mathrm{bl}(\chi_H)$, and we conclude that $\xi \in \mathrm{bl}(\chi_H)$ so $c = \mathrm{bl}(\chi_H)$. □

Corollary 5.9 (Brauer's third main). *Let $H \leqslant G$ and $b \in \mathrm{Bl}(H)$. If b^G is defined and $b^G = B_0(G)$ then $b = B_0(H)$.*

Proof. Apply Okuyama's theorem to $\chi = 1_G$. □

LECTURE 6

Clifford theory for Brauer characters

A lot of the Clifford theory of Brauer characters mimicks the Clifford theory of ordinary characters. The main challenges in the proofs arise from the fact that Frobenius' reciprocity is no longer true and the absence of a scalar product. We will not prove many results that essentially just mimick the techniques of the complex situation with the appropiate adjustments. Every unproven theorem (and the proven ones as well) can be found in [**Nav98**, Chapter 8].

6.1. Induction of Brauer characters and Clifford's theorem

Let $\alpha \in \mathrm{cf}(H^0)$ where $H \subseteq G$. We define $\alpha^G \in \mathrm{cf}(G^0)$ by the formula

$$\alpha^G(x) = \frac{1}{|H|} \sum_{g \in G} \tilde{\alpha}(gxg^{-1})$$

where $\tilde{\alpha}(y) = \alpha(y)$ if $y \in H^0$ and $\tilde{\alpha}(y) = 0$ otherwise. Note that if $\beta \in \mathrm{cf}(H)$ then $(\beta^G)^0 = (\beta^0)^G$.

Theorem 6.1 (Brauer–Nesbitt). *If α is a Brauer character of $H \leqslant G$ then α^G is a Brauer character of G.*

In the above situation, we denote by $\mathrm{IBr}(G|\alpha)$ the set of characters $\chi \in \mathrm{IBr}(G)$ such that χ_H contains α. Warning! This does not imply (I think) that α^G contains χ (we will see below that it does if $H \lhd G$).

One of the main obstacles of proving the above theorem (and many others in this chapter) is the fact that there is no scalar product to help us decompose Brauer characters as a linear combination of irreducible Brauer characters. In fact, for complex characters, the previous proof is just a direct consequence of the Frobenius reciprocity, which we lack in this context. The closest analogue to the scalar product that helps us in this situation is the following. If $\chi = \sum_{\mu \in \mathrm{IBr}(G)} a_\mu \mu$ and $\varphi = \sum_{\mu \in \mathrm{IBr}(G)} b_\mu \mu$ then we define

$$\mathbf{I}(\chi, \varphi) = \sum_{\mu \in \mathrm{IBr}(G)} a_\mu b_\mu$$

and in particular $\mathbf{I}(\chi, \chi) = 1$ if and only if $\chi \in \mathrm{IBr}(G)$.

If $N \lhd G$, $\theta \in \mathrm{IBr}(N)$ and $g \in G$ then we define by θ^g the character defined by

$$\theta^g(n) = \theta(gng^{-1})$$

for $n \in N^0$. This defines an action of G on $\mathrm{IBr}(N)$ by conjugation.

It turns out that the analogue Frobenius reciprocity is true if we assume the subgroup is normal! In fact, Clifford's theorem also holds for Brauer characters.

THEOREM 6.2 (Clifford). *Let $N \lhd G$, let $\varphi \in \mathrm{IBr}(G)$ and $\theta \in \mathrm{IBr}(N)$. Then*

(i) *φ is a constituent of θ^G with multiplicity e if and only if θ is a constituent of φ_N*

(ii) *in this case*

$$\varphi_N = e\sum_{i=1}^{t} \theta_i$$

where $\{\theta_1, \ldots, \theta_t\}$ is the set of G-conjugates of θ and $e = \mathbf{I}(\chi_N, \varphi) = \mathbf{I}(\chi, \varphi^G)$.

A key result is that if V is a simple FG-module and W is a FN-submodule of V_N then $V_N = \sum_{g\in G} Wg$. Most of the proof then follows the proof for complex characters with the appropiate substitutions of the scalar product by our newly defined $\mathbf{I}$.

If $N \lhd G$ and $\theta \in \mathrm{IBr}(G)$ we denote by G_θ the stabilizer of θ in G (also called the inertia subgroup of θ).

THEOREM 6.3 (Clifford correspondence). *Let $N \lhd G$ and $\theta \in \mathrm{IBr}(G)$. Then the map*

$$\mathrm{IBr}(G_\theta|\theta) \to \mathrm{IBr}(G|\theta)$$
$$\psi \mapsto \psi^G$$

is a bijection. Moreover if $\psi \in \mathrm{IBr}(G_\theta|\theta)$ we have

$$\mathbf{I}((\psi^G)_N, \theta) = \mathbf{I}(\psi_N, \theta) \quad \text{and} \quad \mathbf{I}((\psi^G)_{G_\theta}, \psi) = 1.$$

If we compare this to the case of complex characters then the last two conditions might become more natural (recall that in the complex case $(\psi^G)_T = \psi + \Delta$ where Δ is a sum (or zero) of characters not lying over θ).

The following is perhaps surprising (it is to me).

PROPOSITION 6.4. *Let $N \lhd G$ and $\theta \in \mathrm{IBr}(N)$. If $\tau \in \mathrm{IBr}(G_\theta|\theta)$ then $(\Phi_\tau)^G = \Phi_{\tau^G}$.*

6.2. Extendibility of Brauer characters

An extension of a character $\alpha \in \mathrm{IBr}(H)$ is a character $\chi \in \mathrm{IBr}(G)$ with $\chi_H = \alpha$. When $H \lhd G$ there are some situations in which we can guarantee extendibility (and then we have great control of the characters in $\mathrm{IBr}(G|\alpha)$.

THEOREM 6.5 (Green). *Let $N \lhd G$ and assume G/N is a p-group. Let $\theta \in \mathrm{IBr}(N)$. Then there is a unique $\varphi \in \mathrm{IBr}(G|\theta)$ and $\varphi_N = \sum_{i=1}^t \theta_i$ with usual notation. In particular, if θ is G-invariant then $\varphi_N = \theta$.*

PROOF. We argue by induction on $|G : N|$.

First we claim that we may assume θ is G-invariant. Indeed, if $G_\theta < G$ then by induction we have $\mathrm{IBr}(G_\theta|\theta)$ contains a unique character, say ψ. By the Clifford correspondence we have that ψ^G is the unique character in $\mathrm{IBr}(G|\theta)$. Furthermore, by induction $\psi_N = \theta$ so $\mathbf{I}(\psi_N, \theta) = 1$ by Clifford's theorem and again by the Clifford correspondence, $\mathbf{I}((\psi^G)_N, \theta) = 1$. The result follows now by applying Clifford's theorem again.

Therefore we assume $G_\theta = G$, so if $\varphi \in \mathrm{IBr}(G|\theta)$ we have $\varphi_N = e\theta$. Notice that $N^0 = G^0$. By the induction formula,

$$(\theta^G)_N = |G : N|\theta$$

so for any $n \in N^0 = G^0$ we have

$$e\theta^G(n) = e|G : N|\theta(n) = |G : N|\varphi(n)$$

which shows that $e\theta^G = |G : N|\varphi$ so θ^G is a multiple of φ by the linear independence of $\mathrm{IBr}(G)$ (and the uniqueness is proved). We need to show that $e = 1$.

If $e > 1$ then since $\theta^G = \frac{|G:N|}{e}\varphi$ it follows that e is a p-power. In particular, for all $g \in G^0 = N^0$ we have

$$\varphi(g)^* = e^*\theta(g)^* = 0.$$

Now if $x \in G$ and $\mathcal{X}$ affords φ then recall that by Lemma 1.4 we have

$$\mathrm{trace}(\mathcal{X}(x)) = \varphi(x_{p'})^*$$

so $\mathrm{trace}(\mathcal{X}(x)) = 0$ for all $x \in G$. This contradicts the fact that trace functions of irreducible representations are linearly independent, so we conclude $e = 1$. □

There is another classic situation in which extendibility can be guaranteed, but its only known proof uses representations and is a bit annoying.

THEOREM 6.6. *Let $N \lhd G$ and assume G/N is cyclic. If $\theta \in \mathrm{IBr}(N)$ is G-invariant then θ extends to G.*

In these (and the many other) situations where we have a character θ from a normal subgroup N that extends to G, there is a perfect understanding of the set $\mathrm{IBr}(G|\theta)$.

THEOREM 6.7 (Gallagher correspondence). *Let $N \lhd G$ and assume θ extends to $\tilde{\theta} \in \mathrm{IBr}(G)$. Then the map*

$$\mathrm{IBr}(G/N) \to \mathrm{IBr}(G|\theta)$$
$$\beta \mapsto \beta\tilde{\theta}$$

is a bijection.

Another warning! As with ordinary characters, unless a *canonical* extension can be found, there is no *canonical* description of $\mathrm{IBr}(G|\theta)$, i.e., our description will depend on the extension chosen. For most applications this is not very important. Notice also that if G/N is a p-group, then Green's theorem and the Gallagher correspondence coincide since $1_{G/N}$ is the only Brauer character of G/N.

Recall that in general if φ is an irreducible Brauer character of G, it is not even true that $\varphi(1)$ divides $|G|$ (not even $\varphi(1)_p$ divides $|G|$). An example can be found in the sporadic McLaughlin group McL which has a 2-Brauer character φ with $\varphi(1)_2 = 2^9$ but $|\mathrm{McL}|_2 = 2^7$. This cannot happen whenever G is solvable. In fact, more is true:

COROLLARY 6.8 (Swan). *Let $N \lhd G$ and $\theta \in \mathrm{IBr}(N)$. If G/N is solvable then $\varphi(1)/\theta(1)$ divides $|G : N|$ for all $\varphi \in \mathrm{IBr}(G|\theta)$.*

PROOF. Argue by induction on $|G : N|$.

First we claim that we may assume θ is G-invariant. Again, if $\chi \in \mathrm{IBr}(G|\theta)$ and $G_\theta < G$ then by induction the Clifford correspondent ψ of χ over θ satisfies $\psi(1)/\theta(1) \mid |G_\theta : N|$. Since $\psi^G = \chi$ then

$$\chi(1)/\theta(1) = |G : G_\theta|\psi(1)/\theta(1)$$

divides $|G : N| = |G : G_\theta||G_\theta : N|$.

Now if $N \lhd M \lhd G$ and again let $\chi \in \mathrm{IBr}(G|\theta)$ and let $\psi \in \mathrm{IBr}(G|\theta)$ be any constituent of χ_M. By induction, $\psi(1)/\theta(1) \mid |M : N|$ and $\chi \in \mathrm{IBr}(G|\psi)$ satisfies $\chi(1)/\psi(1) \mid |G : M|$. We conclude that $\chi(1)/\theta(1)$ divides $|G : N|$. Therefore we may assume G/N has no proper normal subgroups.

Since G/N is solvable and simple, it is cyclic. By Theorem 6.6 θ extends to G and by the Gallagher correspondence, every $\chi \in \mathrm{IBr}(G|\theta)$ is an extension of θ. Therefore $\chi(1)/\theta(1) = 1$ and we are done. □

The above result is just true for complex characters without any solvability condition.

6.3. On modular character triples

Character triples are one of the fundamental tools to work with Clifford theory. The precise definition is a bit too technical to state here but we will try to give some intuition and explain some of the results that one can obtained by applying this theory.

A modular character triple is a triple (G, N, θ) where θ is a G-invariant Brauer character of $N \lhd G$. It can happen that two finite groups have different Brauer character theories, but that over a particular character of a normal subgroup *the same things* happen. Of course, for this to be possible, the factor groups have to be isomorphic.

We say that two modular character triples $(G, N, \theta), (H, M, \lambda)$ are isomorphic if there is a group isomorphism $\sigma : G/N \to H/M$ and for any subgroup $N \leqslant L \leqslant G$ there is a map

$$\tau_L : \mathbb{N}[\mathrm{IBr}(L|\theta)] \to \mathbb{N}[\mathrm{IBr}(\sigma(L)|\lambda)]$$

that maps irreducible characters to irreducible characters bijectively and satsifies many more compatibility properties (with conjugation, restriction, induction, multiplication... etc). A particular condition is that if $\chi \in \mathrm{IBr}(L|\theta)$ then

$$\frac{\chi(1)}{\theta(1)} = \frac{\tau_L(\chi)(1)}{\lambda(1)}.$$

Also, θ extends to a subgroup $N \leqslant L \leqslant G$ if and only if λ extends to the corresponding subgroup $\sigma(L)$.

THEOREM 6.9. *Any modular character triple (G, N, θ) is isomorphic to a modular character triple (H, M, λ) where $M \leqslant \mathbf{Z}(H)$ and λ is linear and faithful. In particular M has order not divisible by p.*

Notice that the corresponding λ is also an irreducible complex character of M. Essentially, every G-invariant $\theta \in \mathrm{IBr}(N)$ corresponds to some element $\alpha \in \mathrm{H}^2(G/N, F^\times)$ (the second cohomology group), and we use this α to construct a central extension of G that ends up yielding the above isomorphism. The construction is quite technical but allows for very nice control of blocks, thanks to results of Murai.

We do not need any of this though. Here are some results that can be obtained as consequences of this isomorphism. In the following results we use that their complex-character version is true.

PROPOSITION 6.10. *Let $N \lhd G$, $\theta \in \mathrm{IBr}(N)$ and assume G/N is p-solvable. Then for any $\chi \in \mathrm{IBr}(G|\theta)$ we have $\chi(1)/\theta(1)$ divides $|G : N|$.*

PROOF. We argue by induction on $|G : N|$. As in the solvable version, we may assume θ is G-invariant. By Theorem 6.9, (G, N, θ) is isomorphic to (H, M, λ)

where λ is a linear character and M is a p'-group. Notice that it suffices to prove the result for (H, M, λ).

Let $M \lhd U \lhd H$ be a minimal normal subgroup. Since $H/M \cong G/N$ is p-solvable, U/M is either a p-group or a p'-group. If U/M is a p-group then we are done by Theorem 6.5. If U/M is a p'-group then U is a p'-group and $\mathrm{IBr}(U) = \mathrm{Irr}(U)$. The result now follows by using the complex character version. □

Using similar ideas it is possible to also prove the following.

PROPOSITION 6.11. *Let (G, N, θ) be a modular character triple. Then θ extends to G if and only if it extends to Q, where $Q/N \in \mathrm{Syl}_q(G/N)$, for every prime q dividing $|G : N|$.*

It is fundamental to argue by induction on $|G : N|$ instead of $|G|$ when working with character triple isomorphisms. There is no control over the order of the group obtained in Theorem 6.9, but we do know that the index of the corresponding normal subgroup is $|G : N|$.

Character triples (both ordinary and modular) have become a fundamental tool in the study and reduction of the main conjectures that we face.

LECTURE 7

Blocks and normal subgroups

For this chapter, if $B \in \mathrm{Bl}(G)$ then we write $B = \mathrm{Irr}(B) \cup \mathrm{IBr}(B)$.

7.1. Block covering

7.1.1. Actions by automorphisms. The group $\mathrm{Aut}(G)$ acts on $\mathrm{cf}(G)$ and $\mathrm{cf}(G^0)$ by

$$\psi^\sigma(g) = \psi(g^{\sigma^{-1}}).$$

This action restricts to the subsets of ordinary (and Brauer) characters, and also restricts to $\mathrm{Irr}(G)$ and $\mathrm{IBr}(G)$. Further, $(\psi^\sigma)^0 = (\psi^0)^\sigma$ for all $\psi \in \mathrm{cf}(G)$ and it follows that $d_{\chi\psi} = d_{\chi^\sigma\psi^\sigma}$. By using the linking graph it follows that $B^\sigma = \{\psi^\sigma \mid \psi \in B\}$ is also a block of G, so $\mathrm{Aut}(G)$ also acts on $\mathrm{Bl}(G)$.

If A is a ring (say $F, \mathbb{C}$ or $\mathcal{S}$) then every $\sigma \in \mathrm{Aut}(G)$ also induces an automorphism of the A-algebra AG, by

$$\left(\sum_{g\in G} a_g g\right)^\sigma = \sum_{g\in G} a_g g^\sigma$$

which maps $\mathbf{Z}(AG)$ to $\mathbf{Z}(AG)$. This action also satisfies $e_\chi^\sigma = e_{\chi^\sigma}$, and therefore $f_B^\sigma = f_{B^\sigma}$ and $e_B^\sigma = e_{B^\sigma}$.

Now let $B \in \mathrm{Bl}(G)$ and $\sigma \in \mathrm{Aut}(G)$. Then $\lambda_B \circ \sigma^{-1}$ is an algebra homomorphism $\mathbf{Z}(FG) \to F$ and

$$(\lambda_B \circ \sigma^{-1})(e_{B^\sigma}) = \lambda_B((e_B^\sigma)^{\sigma^{-1}}) = \lambda_B(e_B) = 1$$

so $\lambda_{B^\sigma} = \lambda_B \circ \sigma^{-1}$.

PROBLEM 7.1. *Let $B \in \mathrm{Bl}(G)$, $D \in \delta(B)$ and $\sigma \in \mathrm{Aut}(G)$. Prove that D^σ is a defect group B^σ*

7.1.2. Blocks of normal subgroups. Applying the previous discussion to the action of G by conjugation on $N \lhd G$ we obtain a G-action on $\mathrm{Bl}(N)$. Notice that if $b \in \mathrm{Bl}(N)$ and $g \in G$ then

$$\mathrm{Irr}(b^g) = \{\psi^g \mid \psi \in \mathrm{Irr}(b)\} \text{ and } \mathrm{IBr}(b^G) = \{\varphi^g \mid \varphi \in \mathrm{Irr}(b)\}.$$

We denote by G_b the stabilizer of b in G (notice that G_b contains G_ψ for all $\psi \in B$).

Proposition 7.2. *Let $\{b_1, \ldots, b_t\} \subseteq \mathrm{Bl}(N)$ be a G-orbit. Then the idempotent $\sum_{i=1}^t f_{b_i}$ lies in $\mathbf{Z}(\mathcal{S}G)$.*

Proof. We have that $\sum f_{b_i} \in \mathcal{S}N \subseteq \mathcal{S}G$. Now let $g \in G$. Then

$$g^{-1}(\sum f_{b_i})g = \sum f_{b_i^g} = \sum f_{b_i}$$

so $\sum f_{b_i} \in \mathbf{Z}(\mathcal{S}G)$. □

Since $\sum_{i=1}^t f_{b_i}$ is an idempotent in $\mathbf{Z}(\mathcal{S}G) \subseteq \mathbf{Z}(\mathbb{C}G)$ there is some subset $\mathcal{A} := \{\chi_1, \ldots, \chi_n\} \subseteq \mathrm{Irr}(G)$ with

$$f_{\mathcal{A}} = \sum_{j=1}^{n} e_{\chi_j} = \sum_{i=1}^{t} f_{b_i} \in \mathbf{Z}(\mathcal{S}G).$$

By Theorem 2.6 we have that $\mathcal{A}$ must be a union of blocks, so there are $\{B_1, \ldots, B_s\} \subseteq \mathrm{Bl}(G)$ such that

$$\sum_{j=1}^{s} f_{B_j} = \sum_{i=1}^{t} f_{b_i}.$$

Definition 7.3. *In the situation above, we say that B_j covers (any) b_i.*

Write $\mathrm{Bl}(G|b)$ for the set of blocks of G that cover b. If the b_i's above are the G-conjugates of b then

$$\mathrm{Bl}(G|b) = \{B_1, \ldots, B_s\}.$$

Problem 7.4. *Let $\{b_1, \ldots, b_t\}$ be the G-orbit of $b \in \mathrm{Bl}(N)$. Prove that*

$$\sum_{B \in \mathrm{Bl}(G|b)} e_B = \sum_{i=1}^{t} e_{b_i}.$$

We aim to characterize block coverings in terms of the characters in the block.

Problem 7.5. *Let $N \lhd G$ and let K be either F or $\mathbb{C}$. Then $\{\mathfrak{Cl}_G(x)^+ \mid x \in N\}$ is a basis of $\mathbf{Z}(KN) \cap \mathbf{Z}(KG)$.*

Proposition 7.6. *Let $N \lhd G$, let $\chi \in \mathrm{Irr}(G)$ and $\theta \in \mathrm{Irr}(N)$. Then $\chi \in \mathrm{Irr}(G|\theta)$ if and only if $\omega_\chi(\mathfrak{Cl}_G(x)^+) = \omega_\theta(\mathfrak{Cl}_G(x)^+)$ for every $x \in N$.*

Proof. We observe first that $\omega_\theta(\mathfrak{Cl}_G(x)^+) = \omega_\psi(\mathfrak{Cl}_G(x)^+)$ for all $x \in N$ if and only if ψ and θ are G-conjugate. Indeed $(\mathfrak{Cl}_G(x)^+)^g = \mathfrak{Cl}_G(x)^+$ so

$$\omega_{\theta^g}(\mathfrak{Cl}_G(x)^+) = \omega_{\theta^g}((\mathfrak{Cl}_G(x)^+)^g) = \omega_\theta(\mathfrak{Cl}_G(x)^+)$$

which proves the if direction. Conversely, assume that ω_θ and ω_ψ agree in $\mathbf{Z}(\mathbb{C}G) \cap \mathbf{Z}(\mathbb{C}N)$ (by Problem 7.5). If $\{\theta_1, \ldots, \theta_t\}$ are the G-conjugates of θ then arguing as in Proposition 7.2

$$\sum_{i=1}^{t} e_{\theta_i} \in \mathbf{Z}(\mathbb{C}G) \cap \mathbf{Z}(\mathbb{C}N)$$

so using that $\omega_\chi(e_\eta) = \delta_{\chi,\eta}$ we have that

$$0 \neq \omega_\theta \left(\sum e_{\theta_i} \right) = \omega_\psi \left(\sum e_{\theta_i} \right)$$

and again since $\omega_\chi(e_\eta) = \delta_{\chi,\eta}$ we have ψ is one of the θ_i's, as desired.

Now let $\xi \in \mathrm{Irr}(N)$ be under χ and let $\{\xi_1, \ldots, \xi_t\}$ be the set of G-conjugates of ξ. By Clifford's theorem we have

$$\chi_N = e \sum_{i=1}^{t} \xi_i.$$

If $x \in N$ then write $\mathfrak{Cl}_G(x) = \coprod \mathfrak{Cl}_N(x_j)$ as a disjoint union. Then

$$et\xi(1)\omega_\xi(\mathfrak{Cl}_G(x)^+) = e\xi(1) \sum_i \omega_{\xi_i}(\mathfrak{Cl}_G(x)^+) = e\xi(1) \sum_i \sum_j \omega_{\xi_i}(\mathfrak{Cl}_N(x_i)) =$$

$$= e \sum_i \sum_j |\mathfrak{Cl}_N(x_j)| \xi_i(x_j) = \sum_j |\mathfrak{Cl}_N(x_j)| \left(\sum_i e\xi_i(x_j) \right) =$$

$$= \sum_j |\mathfrak{Cl}_N(x_j)| \chi(x_j) = |\mathfrak{Cl}_G(x)| \chi(x) = \chi(1)\omega_\chi(\mathfrak{Cl}_G(x)^+)$$

and we are done because $et\xi(1) = \chi(1)$. □

Recall that we denote $B = \mathrm{Irr}(B) \cup \mathrm{IBr}(B)$.

THEOREM 7.7. *Let $b \in \mathrm{Bl}(N)$ and $B \in \mathrm{Bl}(G)$. The following conditions are equivalent.*

(i) *B covers b,*

(ii) *for all $\chi \in B$, every irreducible constituent of χ_N lies in a G-conjugate of b,*

(iii) *there is $\chi \in B$ such that χ_N has a constituent in b.*

PROOF. We first prove the theorem for ordinary characters. It is clear that (ii) implies (iii).

Write $\{b_1, \ldots, b_t\}$ for the G-conjugates of b and let $\{B_1, \ldots, B_s\} = \mathrm{Bl}(G|b)$, so that

$$\sum_{i=1}^{t} f_{b_i} = \sum_{j=1}^{s} f_{B_j} \in \mathbf{Z}(\mathbb{C}G) \cap \mathbf{Z}(\mathbb{C}N).$$

Assume that $B \in \mathrm{Bl}(G|b)$ and let $\chi \in \mathrm{Irr}(B)$. Let $\theta \in \mathrm{Irr}(N)$ be under χ. By Proposition 7.6,

$$1 = \omega_\chi \left(\sum_{j=1}^{s} f_{B_j} \right) = \omega_\theta \left(\sum_{i=1}^{t} f_{b_i} \right)$$

which implies that θ lies in one of the b_i's. This shows that (i) imples (ii).

Now if (iii) for $\chi \in \mathrm{Irr}(B)$, let θ be under χ and in $\mathrm{Irr}(b)$. Then again by Proposition 7.6 we have

$$1 = \omega_\theta \left(\sum_{i=1}^{t} f_{b_i} \right) = \omega_\chi \left(\sum_{j=1}^{s} f_{B_j} \right)$$

which shows that ω_χ has to lie in one of the B_j's, and we are done.

To prove the result for Brauer characters, notice that if $\chi \in \mathrm{Irr}(B)$ then $(\chi^0)_N = (\chi_N)^0$. Then for $\varphi \in \mathrm{IBr}(B)$ and $\theta \in \mathrm{IBr}(N)$ under χ, we take χ with $d_{\chi\varphi} \neq 0$ and there is some constituent η of χ_N such that $d_{\eta\theta} \neq 0$, so we use the version of the proof for ordinary characters. □

We obtain the following corollary (an analogue of Clifford's theorem).

Corollary 7.8. *If $b_1, b_2 \in \mathrm{Bl}(N)$ are covered by $B \in \mathrm{Bl}(G)$ then b_1 and b_2 are G-conjugate.*

Now we go the other way.

Proposition 7.9. *Let $b \in \mathrm{Bl}(N)$ be covered by B. For all $\theta \in b$ there is $\chi \in B$ lying over θ.*

Proof. Assume first that θ is an ordinary character. Let $\chi \in \mathrm{Irr}(B)$ and $\eta \in \mathrm{Irr}(b)$ be under χ (by Theorem 7.7 this character exists). Now θ and η lie in the same block so they are connected in the linking graph (while perhaps not being linked).

Assume first that η and θ are linked. Let $\varphi \in \mathrm{IBr}(b)$ be such that $d_{\eta\varphi} \neq 0 \neq d_{\theta\varphi}$. Then $(\chi_N)^0$ contains φ. Therefore there is some constituent $\psi \in \mathrm{IBr}(B)$ of χ^0 lying over φ. Now θ^0 also contains φ so we may write

$$(\theta^G)^0 = (\theta^0)^G = \varphi^G + \Delta$$

where Δ is a Brauer character or zero. Now recall that since $N \lhd G$, Clifford's theorem implies that ψ is an irreducible constituent of φ^G, so there is an irreducible constituent $\xi \in \mathrm{Irr}(G)$ of θ^G such that ξ^0 contains ψ. We have that $\xi \in \mathrm{Irr}(B)$ because $\psi \in \mathrm{IBr}(B)$, and ξ lies over θ, and we have shown that there is a character in B lying over θ.

Now if η and θ are not linked, then there are

$$\eta = \tau_1, \ldots, \tau_n = \theta$$

characters of $\mathrm{Irr}(b)$ such that τ_i is linked to τ_{i+1}, and we can argue by induction and apply the previous argument to conclude the theorem.

Finally, if $\theta \in \mathrm{IBr}(b)$ then let $\gamma \in \mathrm{Irr}(b)$ with $d_{\gamma\theta} \neq 0$ and let $\chi \in \mathrm{Irr}(B)$ lie over γ. Then some constituent of χ^0 lies over θ and we are done. □

Corollary 7.10. *Suppose that G/N is a p-group. If $b \in \mathrm{Bl}(N)$ then there is a unique $B \in \mathrm{Bl}(G|b)$.*

Proof. Use Proposition 7.9 and Green's theorem. □

Proposition 7.11. *Let $B \in \mathrm{Bl}(G)$ and $b \in \mathrm{Bl}(N)$. Then B covers b if and only if $\lambda_B(\mathfrak{Cl}_G(x)^+) = \lambda_b(\mathfrak{Cl}_G(x)^+)$ for all $x \in N$.*

Proof. By Problem 7.4 we may write

$$\sum_{B \in \mathrm{Bl}(G|b)} e_B = \sum_{i=1}^{t} e_{b_i}$$

where $\{b_1, \ldots, b_t\}$ is the G-orbit of b.

Assume first that B covers b and let $\theta \in \mathrm{Irr}(b)$ and $\chi \in \mathrm{Irr}(B)$ lying over θ. Then $\lambda_B = \lambda_\chi$ and $\lambda_b = \lambda_\theta$. For all $x \in N$, using Proposition 7.6 we have that

$$\omega_\chi(\mathfrak{Cl}_G(x)^+) = \omega_\theta(\mathfrak{Cl}_G(x)^+)$$

so

$$\lambda_B(\mathfrak{Cl}_G(x)^+) = \omega_\chi(\mathfrak{Cl}_G(x)^+)^* = \omega_\theta(\mathfrak{Cl}_G(x)^+)^* = \lambda_b(\mathfrak{Cl}_G(x)^+).$$

Conversely suppose that $\lambda_B(\mathfrak{Cl}_G(x)^+) = \lambda_b(\mathfrak{Cl}_G(x)^+)$ for all $x \in N$. Since $\sum_{B \in \mathrm{Bl}(G|b)} e_B = \sum_{i=1}^{t} e_{b_i} \in \mathbf{Z}(FG) \cap \mathbf{Z}(FN)$, by Problem 7.5 we have that

$$\lambda_B\left(\sum_{B \in \mathrm{Bl}(G|b)} e_B\right) = \lambda_b\left(\sum_{i=1}^{t} e_{b_i}\right) = 1$$

because b is one of the b_i's. It follows that B is one of the blocks in $\mathrm{Bl}(G|b)$. □

7.2. The Fong–Reynolds correspondence

The Fong–Reynolds correspondence is the analogue of the Clifford correspondence for blocks.

Theorem 7.12. *Let $b \in \mathrm{Bl}(N)$.*

(i) *The map*

$$\begin{aligned}\mathrm{Bl}(G_b|b) &\to \mathrm{Bl}(G|b)\\ B &\mapsto B^G\end{aligned}$$

is a bijection.

(ii) *If* $B \in \mathrm{Bl}(G_b|b)$ *then*

$$\mathrm{Irr}(B^G) = \{\psi^G \mid \psi \in \mathrm{Irr}(B)\} \text{ and } \mathrm{IBr}(B^G) = \{\varphi^G \mid \varphi \in \mathrm{IBr}(B)\}.$$

(iii) *Every defect group of* B *is a defect group of* B^G.

(iv) *If* $\chi \in \mathrm{Irr}(B)$ *then* $h_\chi = h_{\chi^G}$.

Sketch of proof Recall that for any $\theta \in b$ we have $G_\theta \subseteq G_b$. Let $B \in \mathrm{Bl}(G_b|b)$ and $\psi \in \mathrm{Irr}(B)$ (the argument for Brauer characters is the same). Then there is an irreducible constituent θ of ψ_N in b. By the Clifford correspondence (for ordinary characters), $\psi = \eta^{G_b}$ for some $\eta \in \mathrm{Irr}(G_\theta|\theta)$. Thus $\psi^G = \eta^G \in \mathrm{Irr}(G|\theta)$ again by the Clifford correspondence. By Corollary 5.3, B^G is defined and contains ψ^G, and B^G also covers b (because ψ^G lies over θ). We have shown that every $\psi \in \mathrm{Irr}(B)$ induces irreducibly to a character in B^G.

Now if $\varphi \in \mathrm{IBr}(B)$ we can argue as before to show $\varphi^G \in \mathrm{IBr}(G)$. We want to show that φ^G is also in B^G. Let $\theta \in \mathrm{IBr}(N)$ lie under φ and let $\eta \in \mathrm{IBr}(G_\theta)$ be its Clifford correspondent, so that $\eta^G = \varphi^G$. Now by Proposition 6.4 we have $\Phi_\varphi = (\Phi_\eta)^{G_b}$. Then

$$\Phi_\varphi^G = ((\Phi_\eta)^{G_b})^G = (\Phi_\eta)^G = \Phi_{\eta^G} = \Phi_{\varphi^G}.$$

Now

$$\Phi_{\varphi^G} = \Phi_\varphi^G = \sum_{\mu \in \mathrm{Irr}(B)} d_{\mu\varphi}\mu^G$$

and every $\mu^G \in \mathrm{Irr}(B^G)$, so $d_{\mu^G\varphi^G} \neq 0$ for some $\mu^G \in \mathrm{Irr}(B^G)$, so $\varphi^G \in \mathrm{IBr}(B^G)$.

With these ideas one can end up showing that this thing is indeed a bijection (notice that we have not proven injectivity nor surjectivity) and conclude (i) and (ii) (the decomposition numbers by arguing as before with the projective indecomposable character).

For the defect groups, notice that by Lemma 4.4 a defect group D of B is contained in a defect group Q of B^G. Now

$$\begin{aligned}|G:Q|_p &= \min\{\psi^G(1)_p \mid \psi \in \mathrm{Irr}(B)\} = |G:G_b|_p \min\{\psi(1)_p \mid \psi \in \mathrm{Irr}(B)\} =\\ &= |G:G_b|_p |G_b:D|_p = |G:D|_p\end{aligned}$$

and we conclude that $D = Q$.

Finally the heights. Let $B \in \mathrm{Bl}(G_b|b)$ and $\psi \in \mathrm{Irr}(B)$ and let D be a defect group of B. Then by the definition of height

$$\nu(\psi(1)) = \nu(|G_b|) - \nu(|D|) + h_\psi$$

so

$$\begin{aligned}\nu(|G|) - \nu(|D|) + h_{\psi^G} = \nu(\psi^G(1)) = \nu(|G : G_b|) + \nu(\psi(1)) = \\ = \nu(|G : G_b|) + \nu(|G_b|) - \nu(|D|) + h_\psi = \nu(|G|) - \nu(|D|) + h_\psi\end{aligned}$$

as desired. □

A big warning: even if we have a defect group of B^G contained in G_b, it is not necessarily a defect group of B!

LECTURE 8

Blocks and normal subgroups II

8.1. Block domination

Recall that if $N \lhd G$, we identify $\mathrm{Irr}(G/N)$ and $\mathrm{IBr}(G/N)$ as subsets of characters in $\mathrm{Irr}(G)$ and $\mathrm{IBr}(G)$ containing N in their kernel via $\chi(x) \mapsto \overline{\chi}(xN)$. Notice that if $x \in G^0$ then

$$\sum_{\varphi \in \mathrm{IBr}(G)} d_{\chi\varphi}\varphi(x) = \chi(x) = \overline{\chi}(xN) = \sum_{\overline{\eta} \in \mathrm{IBr}(G/N)} d_{\overline{\chi}\overline{\eta}}\overline{\eta}(xN) = \sum_{\overline{\eta} \in \mathrm{IBr}(G/N)} d_{\overline{\chi}\overline{\eta}}\eta(x)$$

and since $\mathrm{IBr}(G)$ is a linearly independent set, we conclude that $d_{\overline{\chi}\overline{\eta}} = d_{\chi\eta}$. There are two consequences.

Corollary 8.1. *If $\chi \in \mathrm{Irr}(G)$ contains N in its kernel, then so does every $\varphi \in \mathrm{IBr}(G)$ with $d_{\chi\varphi} \neq 0$.*

Corollary 8.2. *If $\overline{B} \in \mathrm{Bl}(G/N)$ then there exists a unique block $B \in \mathrm{Bl}(G)$ with $\overline{B} \subseteq B$.*

Definition 8.3. *We say $B \in \mathrm{Bl}(G)$ dominates $\overline{B} \in \mathrm{Bl}(G/N)$ if $\overline{B} \subseteq B$.*

Write $\overline{G} = G/N$. Then for any ring A we have a natural algebra homomorphism

$$: AG \to A\overline{G}$$
$$\sum a_g g \mapsto \sum a_g gN$$

Notice that $\overline{e_B}$ is either 0 or it is a central idempotent of $F\overline{G}$. If $\overline{e_B} \neq 0$ then there exist $\overline{B_1}, \ldots, \overline{B_t} \in \mathrm{Bl}(\overline{G})$ such that

$$\overline{e_B} = e_{\overline{B}_1} + \cdots + e_{\overline{B}_t}.$$

Proposition 8.4. *The block B dominates $\overline{B}$ if and only if $\overline{e_B}$ contains $e_{\overline{B}}$ (if and only if $\lambda_{\overline{B}}(\overline{e_B}) = 1$).*

Proof. Under the natural homomorphism $\mathbb{C}G \to \mathbb{C}\overline{G}$ we have that $e_\chi \mapsto e_{\overline{\chi}}$ as long as $\chi \in \mathrm{Irr}(G/N)$ and $\overline{e_\chi} = 0$ otherwise.

Now let $\overline{\chi} \in \mathrm{Irr}(\overline{B})$. We have that $\lambda_{\overline{B}}(\overline{e_B}) = 1$ if and only if

$$0 \neq \omega_{\overline{\chi}}(\sum_{\psi \in \mathrm{Irr}(B)} \overline{e_\psi})^* = \omega_{\overline{\chi}}(\sum_{\substack{\psi \in \mathrm{Irr}(B) \\ N \subseteq \ker(\psi)}} e_{\overline{\psi}})^*$$

which happens if and only if $\overline{\chi}$ is one of the $\overline{\psi} \in \mathrm{Irr}(B)$ with $N \subseteq \ker(\psi)$. This happens if and only if $\overline{B} \subseteq B$. □

Notice that if B dominates $\overline{B}$ then for $z \in \mathbf{Z}(FG)$ we have

$$\lambda_B(z) = \lambda_{\overline{B}}(\overline{z})$$

(this is because the composition of and $\lambda_{\overline{B}}$ is an homomorphism $\mathbf{Z}(FG) \to F$, so it must be λ_B.)

Theorem 8.5. *Let $N \lhd G$ and write $\overline{G} = G/N$.*

(i) *If $\overline{B} \subseteq B \in \mathrm{Bl}(G)$ where $\overline{B} \in \mathrm{Bl}(\overline{G})$, then for any $\overline{D} \in \delta(\overline{B})$ there is $P \in \delta(B)$ with $\overline{D} \subseteq PN/N$.*

(ii) *If N is a p-group then for any block $B \in \mathrm{Bl}(G)$ there is $\overline{B} \in \mathrm{Bl}(\overline{G})$ with $\overline{B} \subseteq B$, and $\delta(\overline{B}) = \{P/N \mid P \in \delta(B)\}$.*

(iii) *If N is a p'-group and $\overline{B} \subseteq B$ then $\mathrm{Irr}(B) = \mathrm{Irr}(\overline{B})$, $\mathrm{IBr}(\overline{B}) = \mathrm{IBr}(B)$ and $\delta(\overline{B}) = \{PN/N \mid P \in \delta(B)\}$*

Proof. Let $\mathfrak{Cl}_G(x)$ be a defect class for B, so that $\mathfrak{Cl}_{\overline{G}}(\overline{x})$ is the conjugacy class of $\overline{x} = xN$ in $\overline{G}$. Write $\mathfrak{Cl}_{\overline{G}}(\overline{x}) = \{\overline{x_1}, \ldots, \overline{x_s}\}$ and notice that

$$\mathfrak{Cl}_G(x) = \coprod_{i=1}^{s} (\mathfrak{Cl}_G(x) \cap x_i N).$$

Furthermore, if $g \in G$ then

$$(\mathfrak{Cl}_G(x) \cap xN)^g = \mathfrak{Cl}_G(x) \cap x^g N$$

and it follows that

$$t := |\mathfrak{Cl}_G(x) \cap xN| = |\mathfrak{Cl}_G(x) \cap x^g N|$$

and it follows that $|\mathfrak{Cl}_G(x)| = t|\mathfrak{Cl}_{\overline{G}}(\overline{x})|$, so

$$\overline{\mathfrak{Cl}_G(x)^+} = t\mathfrak{Cl}_{\overline{G}}(\overline{x})^+.$$

Write $C/N = \mathbf{C}_{\overline{G}}(\overline{x})$, so that $|G : C| = |\mathfrak{Cl}_{\overline{G}}(\overline{x})|$. Then

$$t = \frac{|\mathfrak{Cl}_G(x)|}{|\mathfrak{Cl}_{\overline{G}}(\overline{x})|} = |C : \mathbf{C}_G(x)|.$$

Since $\mathfrak{Cl}_G(x)$ is a defect class for B we have $\lambda_B(\mathfrak{Cl}_G(x)^+) \neq 0$. Using that B dominates $\overline{B}$,

$$0 \neq \lambda_B(\mathfrak{Cl}_G(x)^+) = \lambda_{\overline{B}}(\overline{\mathfrak{Cl}_G(x)^+}) = \lambda_{\overline{B}}(t\mathfrak{Cl}_{\overline{G}}(\overline{x})^+) = t^* \lambda_{\overline{B}}(\mathfrak{Cl}_{\overline{G}}(\overline{x})^+)$$

and it follows that $t \neq 0 \bmod p$. In particular, a Sylow p-subgroup of $\mathbf{C}_G(x)$ is a Sylow p-subgroup of C. Further, since $\lambda_{\overline{B}}(\mathfrak{Cl}_{\overline{G}}(\overline{x})^+) \neq 0$ we have that $\overline{D} \subseteq \overline{X} \in \mathrm{Syl}_p(C/N)$ by the Min–Max theorem. By the Sylow theorems and the previous comment, there is $P \in \mathrm{Syl}_p(\mathbf{C}_G(x))$ with $\overline{D} \subseteq \overline{X} \subseteq PN/N$ and part (a) is proved.

For part (b) notice that N is contained in the kernel of every Brauer character, so there are blocks $\overline{B}_1, \ldots, \overline{B}_t$ of $\overline{G}$ with

$$\mathrm{IBr}(B) = \coprod_{i=1}^{t} \mathrm{IBr}(B_i).$$

We can now use part (a) and the fact that

$$\nu(|G|) - \mathrm{d}(B) = \min\{\nu(\varphi(1)) \mid \varphi \in \mathrm{IBr}(B)\}$$

(see Problem 3.6) to get the result on the defect groups.

For part (c) notice that if $\overline{B} \subseteq B$ then if $\chi \in \mathrm{Irr}(\overline{B})$, viewed as a character of B it lies over 1_N and therefore B covers $\mathrm{bl}(1_N)$ which only contains the ordinary (and modular) character 1_N. It follows that every $\chi \in B$ contains N in its kernel and the result follows. □

Observe that the difference between cases (ii) and (iii) is that when N is a p-group, B is guaranteed to dominate a block of G/N.

8.2. Blocks of $P\mathbf{C}_G(P)$

LEMMA 8.6. *Assume $b \in \mathrm{Bl}(N)$ is such that b^G is defined. Then b^G covers b.*

PROOF. By Proposition 7.11 we only have to check that λ_b and λ_b^G coincide in $\mathfrak{Cl}_G(x)^+$ for $x \in N$. Now

$$\lambda_b^G(\mathfrak{Cl}_G(x)^+) = \lambda_b((\mathfrak{Cl}_G(x) \cap N)^+) = \lambda_b(\mathfrak{Cl}_G(x)^+)$$

and we are done. □

For the next result we need a property of induced blocks which is left as a problem.

PROBLEM 8.7. *Let $K \leqslant H \leqslant G$. Let $b \in \mathrm{Bl}(K)$ and assume b^H is defined. Then b^G is defined if and only if $(b^H)^G$ is defined. In this case, $b^G = (b^H)^G$.*

THEOREM 8.8 (Extended first main theorem). *If $B \in \mathrm{Bl}(G|D)$ then there is a unique $\mathbf{N}_G(D)$-orbit of blocks of $D\mathbf{C}_G(D)$ inducing B, and all of them have defect group D. Moreover, if b is such a block, $b^{\mathbf{N}_G(D)}$ is the Brauer correspondent of B.*

Proof. Let $B \in \mathrm{Bl}(G|D)$. By Theorem 4.5 we know there is $b \in \mathrm{Bl}(D\mathbf{C}_G(D))$ with $b^G = B$. Since $D \lhd D\mathbf{C}_G(D)$ we have that D is contained in a defect group P of b. Since $b^{\mathbf{N}_G(P)}$ is also defined by the same result, we have that P si contained in a defect group Q of $b^{\mathbf{N}_G(D)}$. By the previous problem, $(b^{\mathbf{N}_G(P)})^G = b^G = B$ and therefore Q is contained in a G-conjugate of D. We conclude that b and $b^{\mathbf{N}_G(D)}$ have defect group D. It follows that $b^{\mathbf{N}_G(D)}$ is the Brauer correspondent of B.

It remains to prove that all these blocks are $\mathbf{N}_G(D)$-conjugate. Let $b_1, b_2 \in \mathrm{Bl}(D\mathbf{C}_G(D))$ inducing B, so that $b_1^{\mathbf{N}_G(D)} = b_2^{\mathbf{N}_G(D)}$ is the Brauer correspondent of B. By Lemma 8.6 $b^{\mathbf{N}_G(D)}$ covers b_1 and b_2 so they are $\mathbf{N}_G(D)$-conjugate. □

The blocks $b \in \mathrm{Bl}(D\mathbf{C}_G(D))$ inducing B are called **roots** of B. The remaining part of this lecture consists on studying their structure. Unfortunately, we omit the proof of the following refinement of Theorem 8.5 in this special case.

Proposition 8.9. *Assume G has a normal p-subgroup P and $G/\mathbf{C}_G(P)$ is a p-group. Write $\overline{G} = G/P$. The map $\mathrm{Bl}(\overline{G}) \to \mathrm{Bl}(G)$ defined by $\overline{B} \mapsto B$ if $\overline{B} \subseteq B$ is a bijection. Moreover, $C_B = |P|C_{\overline{B}}$.*

We need to invoke a result which is a consequence of Brauer's second main theorem, which we have not even stated. I am pretty sure that, for normal defect groups, there has to be an easier proof.

Theorem 8.10. *Let $\chi \in \mathrm{Irr}(B)$ where $B \in \mathrm{Bl}(G)$. If g_p is not contained in any defect group of B then $\chi(g) = 0$.*

Proof. See [**Nav98**, Theorem 5.9]. □

The following is an example of what is known as a **nilpotent** block. These were defined by Broué and Puig in a very influential paper, by extending the classical Frobenius theorem on normal p-complement for blocks. The case of the following theorem is a very particular case of this type of blocks.

Theorem 8.11. *Let P be a p-subgroup of G, let $B \in \mathrm{Bl}(G|P)$, and assume $G = \mathbf{C}_G(P)P$. The following hold.*

(i) *There is a unique $\theta \in \mathrm{Irr}(B)$ with $P \subseteq \ker(\theta)$, and in fact*

$$\theta(1)_p = |G : P|_p.$$

(ii) $\mathrm{IBr}(B) = \{\theta^0\}$.

(iii) *The map*

$$\begin{aligned}\mathrm{Irr}(P) &\to \mathrm{Irr}(B)\\ \xi &\mapsto \theta_\xi\end{aligned}$$

is a bijection, where θ_ξ *is deifned as* $\theta_\xi(g) = \theta(g_{p'})\xi(g_p)$ *if* $g_p \in P$ *and* $\theta_\xi(g) = 0$ *otherwise.*

Proof. Let $\overline{B}$ be the unique block of G/P dominated by B. By Proposition 8.9, $\overline{B}$ has trivial defect group, so it has defect zero. Thus $\mathrm{Irr}(\overline{B}) = \{\theta\}$ and $\mathrm{IBr}(B) = \{\theta^0\}$. and the first two parts are proved.

Notice that G acts trivially on $\mathrm{Irr}(P)$, because $G = P\mathbf{C}_G(P)$. By Clifford's theorem, if $\chi \in \mathrm{Irr}(P)$ then we may write $\chi_P = e\xi$ for some $\xi \in \mathrm{Irr}(P)$. Since P is the unique defect group of B, by Theorem 8.10 we have $\chi(g) = 0$ if $g_p \notin P$.

We consider now the case $g_p \in P$. Since $G/\mathbf{C}_G(P)$ is a p-group, every p-regular element is contained in $\mathbf{C}_G(P)$ so it commutes with P. Thus $H := \langle g_{p'}, P\rangle = \langle g_{p'}\rangle \times P$. Write

$$\chi_H = \sum_{\psi\in\mathrm{Irr}(H)} a_\psi \psi.$$

Since $\chi_P = e\xi$ then the constituents of χ_H are of the form $\lambda \times \xi$, so we can write

$$\chi_H = \alpha \times \xi$$

where $\alpha = e\chi_{\langle g_{p'}\rangle}$. This implies that $\chi(g_{p'}) = \alpha(g_{p'})\xi(1)$. Also, since $g \in H$,

$$\chi(g) = \alpha(g_{p'})\xi(g_p) = \frac{\chi(g_{p'})}{\xi(1)}\xi(g_p).$$

Now recall that θ^0 is the unique Brauer character of B. It follows that $\chi(g_{p'}) = d_{\chi\theta^0}\theta(g_{p'})$ where $d_{\chi\theta^0} = \chi(1)/\theta(1)$, so

$$\chi(g) = \frac{d_{\chi\theta^0}}{\xi(1)}\theta(g_{p'})\xi(g_p). \tag{8.2.1}$$

We now need to use two easy to prove group theoretical facts which we leave to the reader. First, since $G^0 \subseteq \mathbf{C}_G(P)$ we have that the map

$$\begin{aligned}G^0 \times P &\to \{g \in G \mid g_p \in P\}\\ (x, y) &\mapsto xy\end{aligned}$$

is a bijection. The map

$$\begin{aligned}G^0 &\to (G/P)^0\\ x &\mapsto xP\end{aligned}$$

is also a bijection.

Using the first bijection, and that $\chi(g) = 0$ if $g_p \notin P$ we compute

$$[\chi,\chi] = \frac{1}{|G|}\sum_{g\in G}\chi(g)\chi(g^{-1}) = \frac{1}{|G|}\sum_{\substack{g\in G\\ g_p\in P}}\chi(g)\chi(g^{-1}) =$$

$$= \left(\frac{d_{\chi\theta^0}}{\xi(1)}\right)^2 \frac{1}{|G|}\sum_{x\in P}\sum_{y\in G^0}\theta(y)\theta(y^{-1})\xi(x)\xi(x^{-1}) =$$

$$= \left(\frac{d_{\chi\theta^0}}{\xi(1)}\right)^2 \frac{1}{|G|}\sum_{y\in G^0}\theta(y)\theta(y^{-1})\left(\sum_{x\in P}\xi(x)\xi(x^{-1})\right) =$$

$$= \left(\frac{d_{\chi\theta^0}}{\xi(1)}\right)^2 \frac{1}{|G:P|}\sum_{y\in G^0}\theta(y)\theta(y^{-1}).$$

Recall that θ has defect zero as a character of G/P, so it vanishes in the p-singular conjugacy classes of G/P. Thus

$$\frac{1}{|G:P|}\sum_{y\in G^0}\theta(y)\theta(y^{-1}) = \frac{1}{|\overline{G}|}\sum_{\overline{y}\in\overline{G}^0}\theta(\overline{y})\theta(\overline{y}^{-1}) = \frac{1}{|\overline{G}|}\sum_{\overline{y}\in\overline{G}}\theta(\overline{y})\theta(\overline{y}^{-1}) =$$

$$= [\theta,\theta] = 1$$

so rewritting the above expression we obtain $1 = [\chi,\chi] = \left(\frac{d_{\chi\theta^0}}{\xi(1)}\right)^2$ so

$$\chi(1)/\theta(1) = d_{\chi\theta^0} = \xi(1).$$

By going back to Equation 8.2.1 we see that $\chi(g) = \theta(g_{p'})\xi_\chi(g_p)$ for a uniquely determined $\xi_\chi \in \mathrm{Irr}(P)$. It remains to show that $\chi \mapsto \xi_\chi$ is a bijection. The fact that it is injective is immediate (χ is totally determined by θ and ξ).

Using again that θ has defect zero as a character of G/P, the Cartan matrix of $\mathrm{bl}(\theta) \in \mathrm{Bl}(G/P)$ is just (1). By using the last part of Proposition 8.9, the cartan matrix of B is $(|P|)$. Thus

$$|P| = \sum_{\chi\in\mathrm{Irr}(B)}(d_{\chi\theta^0})^2 = \sum_{\chi\in\mathrm{Irr}(B)}\xi_\chi(1)^2 \leqslant \sum_{\tau\in\mathrm{Irr}(P)} = |P|$$

and it follows that the map is surjective. □

The character θ from the previous proposition is known as the **canonical** character for B, and it is determined up to $\mathbf{N}_G(D)$-conjugation.

COROLLARY 8.12. *In the above situation, the set of heights of B is $\{\nu(\xi(1)) \mid \xi \in \mathrm{Irr}(P)\}$.*

LECTURE 9

Blocks and normal subgroups III

For the final lecture of the course, we prove Brauer's height zero conjecture for blocks with normal defect group, and we use this as an excuse to introduce even more results on block coverings. A bit of the history and lore around the conjecture will be introduced in Section 9.3.

9.1. Covering blocks and defect groups

We begin by proving some results of Fong on the relation between the defect groups of blocks of G and the blocks they cover in some $N \lhd G$.

Recall that if $b \in \mathrm{Bl}(N)$ then by Problem 7.4

$$\sum_{B' \in \mathrm{Bl}(G|b)} e_{B'} = \sum_{i=1}^{t} e_{b_i}$$

where $\{e_{b_1}, \ldots, e_{b_t}\}$ is the set of G-conjugates of b. Now the elements e_{b_i} are linear combinations of $\mathfrak{Cl}_N(x)^+$ for $x \in G$, but since the elements $e_{B'}$ are sums of $\mathfrak{Cl}_G(y)$ for $y \in G$, it follows that we may write

$$\sum_{B' \in \mathrm{Bl}(G|b)} e_{B'} = \sum_{\substack{x \in G/\sim \\ x \in N}} u_b(x) \mathfrak{Cl}_G(x)^+.$$

Proposition 9.1. *Let $b \in \mathrm{Bl}(N)$ and let $B \in \mathrm{Bl}(G|b)$. Write*

$$\sum_{B' \in \mathrm{Bl}(G|b)} e_{B'} = \sum_{\substack{x \in G/\sim \\ x \in N}} u_b(x) \mathfrak{Cl}_G(x)^+$$

as before. Then there is some $x \in N$ with $u_b(x) \neq 0 \neq \lambda_B(\mathfrak{Cl}_G(x)^+)$. Moreover if $\mathrm{d}(B) \geqslant \mathrm{d}(B')$ for all $B' \in \mathrm{Bl}(G|b)$ then $\delta(\mathfrak{Cl}_G(x)) = \delta(B)$.

Proof. We have

$$1 = \lambda_B \left(\sum_{B' \in \mathrm{Bl}(G|b)} e_{B'} \right) = \lambda_B \left(\sum_{\substack{x \in G/\sim \\ x \in N}} u_b(x) \mathfrak{Cl}_G(x)^+ \right)$$

so the first part follows.

Assume now that $\mathrm{d}(B) \geqslant \mathrm{d}(B')$ for all $B' \in \mathrm{Bl}(G|b)$. Since $\lambda_B(\mathfrak{Cl}_G(x)^+) \neq 0$, by the Min–Max theorem we have that if $D \in \delta(\mathfrak{Cl}_G(x))$, there is some $P \in \delta(B)$ with $P \subseteq D$. Now, since

$$e_{B'} = \sum_{x \in G/\sim} a_{B'}(\mathfrak{Cl}_G(x))\mathfrak{Cl}_G(x)^+$$

so

$$\sum_{B' \in \mathrm{Bl}(G|b)} e_{B'} = \sum_{B' \in \mathrm{Bl}(G|b)} \sum_{x \in G/\sim} a_{B'}(\mathfrak{Cl}_G(x))\mathfrak{Cl}_G(x)^+ = \\ = \sum_{x \in G/\sim} \sum_{B' \in \mathrm{Bl}(G|b)} a_{B'}(\mathfrak{Cl}_G(x))\mathfrak{Cl}_G(x)^+$$

it follows that

$$u_b(x) = \sum_{B' \in \mathrm{Bl}(G|b)} a_{B'}(\mathfrak{Cl}_G(x))$$

so there is some $B_0 \in \mathrm{Bl}(G|b)$ with

$$a_{B_0}(\mathfrak{Cl}_G(x)) \neq 0$$

and from the Min–Max theorem it follows that

$$P \subseteq D \subseteq P_0 \in \delta(B_0).$$

Since $\mathrm{d}(B) \geqslant \mathrm{d}(B_0)$ this implies $|P| \geqslant |P_0|$ which forces $P = D = P_0$, and $D \in \delta(B)$. □

THEOREM 9.2 (Fong). *Let $b \in \mathrm{Bl}(N)$ be G-invariant and assume $B \in \mathrm{Bl}(G|b)$ is such that $\mathrm{d}(B) \geqslant \mathrm{d}(B')$ for all $B' \in \mathrm{Bl}(G|b)$. If $P \in \delta(B)$ we have that p does not divide $|G : PN|$ and $P \cap N \in \delta(b)$.*

PROOF. First notice that, arguing as before,

$$e_b = \sum_{B' \in \mathrm{Bl}(G|b)} e_{B'} = \sum_{\substack{x/\sim_G \\ x \in N}} u_b(x)\mathfrak{Cl}_G(x)^+.$$

Notice further that

$$e_b = \sum_{y \in N/\sim} a_b(\mathfrak{Cl}_N(y))\mathfrak{Cl}_N(y)^+$$

and it follows that $u_b(y) = a_b(\mathfrak{Cl}_N(y))$ for all $y \in N$.

By Proposition 9.1, there is some $x \in N$ with $u_b(x) \neq 0 \neq \lambda_B(\mathfrak{Cl}_G(x)^+)$ and $P \in \delta(\mathfrak{Cl}_G(x))$. Now

$$\mathfrak{Cl}_G(x) = \bigcup_{g \in G} \mathfrak{Cl}_N(x)^g$$

so $\mathfrak{Cl}_G(x)$ is the union of a t different G-conjugates of $\mathfrak{Cl}_N(x)$. More precisely,

$$t = \frac{|\mathfrak{Cl}_G(x)|}{|\mathfrak{Cl}_N(x)|} = \frac{|G||\mathbf{C}_N(x)|}{|N||\mathbf{C}_G(x)|} = \frac{|G||\mathbf{C}_N(x)|}{|N\mathbf{C}_G(x)||\mathbf{C}_G(x) \cap N|} = |G : N\mathbf{C}_G(x)|.$$

Recall that $\lambda_{b^g}(\mathfrak{Cl}_N(x)^+) = \lambda_b((\mathfrak{Cl}_N(x)^+)^g)$. Since b is G-invariant, it follows that $\lambda_b(\mathfrak{Cl}_N(x)^+) = \lambda_b((\mathfrak{Cl}_N(x)^+)^g)$. By Passman (Proposition 7.11)

$$0 \neq \lambda_B(\mathfrak{Cl}_G(x)^+) = \lambda_b(\mathfrak{Cl}_G(x)^+) = t^* \lambda_b(\mathfrak{Cl}_N(x)^+)$$

so we deduce that p does not divide $t = |G : N\mathbf{C}_G(x)|$. Since $P \in \mathrm{Syl}_p(\mathbf{C}_G(x))$, it follows that p does not divide $|G : NP|$. Furthermore,

$$a_b(\mathfrak{Cl}_N(x)) = u_b(x) \neq 0$$

and it follows that

$$a_b(\mathfrak{Cl}_N(x)) \neq 0 \neq \lambda_b(\mathfrak{Cl}_N(x)^+)$$

so $\mathfrak{Cl}_N(x)$ is a defect class for b. Now $\mathbf{C}_G(x) \cap N = \mathbf{C}_N(x)$ and $P \cap N \in \mathrm{Syl}_p(\mathbf{C}_N(x))$ so $P \cap N \in \delta(b)$. □

The fact that $P \cap N \in \delta(b)$ from the previous theorem is in fact true even if we do not assume b is G-invariant. This is a (harder to prove) theorem of Knörr [**Nav98**, Theorem 9.26].

9.2. Regular blocks

We say a block $B \in \mathrm{Bl}(G)$ is regular with respect to $N \lhd G$ if $\lambda_B(\mathfrak{Cl}_G(x)^+) = 0$ for every $x \notin N$. Notice that this does not involve any blocks of N!

PROPOSITION 9.3. *Assume $B \in \mathrm{Bl}(G)$ covers $b \in \mathrm{Bl}(N)$. Then B is regular with respect to N if and only if b^G is defined and $b^G = B$.*

PROOF. Assume b^G is defined and $b^G = B$. If $x \in G\backslash N$ then $\mathfrak{Cl}_G(x) \cap N = \varnothing$. We have

$$\lambda_b^G(\mathfrak{Cl}_G(x)^+) = \lambda_b((\mathfrak{Cl}_G(x) \cap N)^+) = 0$$

and since $B = b^G$ we have $\lambda_B(\mathfrak{Cl}_G(x)^+) = 0$ for all $x \in G\backslash N$. By definition, B is regular with respect to N.

Assume now that B is regular. If $x \in G\backslash N$ then it follows arguing as before that $\lambda_b^G(\mathfrak{Cl}_G(x)^+) = \lambda_B(\mathfrak{Cl}_G(x)^+) = 0$. If $x \in N$ then by Passman (Proposition 7.11) we have

$$\lambda_b^G(\mathfrak{Cl}_G(x)^+) = \lambda_b((\mathfrak{Cl}_G(x) \cap N)^+) = \lambda_b(\mathfrak{Cl}_G(x)^+) = \lambda_B(\mathfrak{Cl}_G(x)^+)$$

and we conclude $\lambda_b^G = \lambda_B$, so we are done. □

PROPOSITION 9.4. *Let $B \in \mathrm{Bl}(G|D)$. If $\mathbf{C}_G(D) \subseteq N$ then B is regular with respect to N.*

PROOF. Let $x \in G$ and assume $\lambda_B(\mathfrak{Cl}_G(x)^+) \neq 0$. By the Min–Max theorem we have that $D \subseteq P \in \delta(\mathfrak{Cl}_G(x))$ which means that $D \subseteq \mathbf{C}_G(x)$, which implies $x \in \mathbf{C}_G(D)$. By hypothesis, $x \in N$, so λ_B must vanish in the conjugacy classes outside N. □

Corollary 9.5. *Let $Q \lhd G$ be a p-subgroup and let $b \in \mathrm{Bl}(Q\mathbf{C}_G(Q))$. Then b^G is the unique block of G which covers b.*

Proof. We know from Theorem 4.5 that b^G is defined, and from Lemma 8.6 that b^G covers b. Let $B \in \mathrm{Bl}(G|b)$. Since $Q \lhd G$ it follows that $Q \subseteq \mathbf{O}_p(G) \subseteq D \in \delta(B)$. This implies that $\mathbf{C}_G(D) \subseteq \mathbf{C}_G(Q) \subseteq Q\mathbf{C}_G(Q)$ and by Proposition 9.4, B is regular with respect to $Q\mathbf{C}_G(Q)$. Now Proposition 9.3 implies that $b^G = B$ as desired. □

Theorem 9.6. *Let $b \in \mathrm{Bl}(D\mathbf{C}_G(D)|D)$ and let T be the stabilizer of b in $\mathbf{N}_G(D)$. Then b^G has defect group D if and only if $|T : D\mathbf{C}_G(D)|$ is not divisible by p.*

Proof. First, we claim that we may assume $D \lhd G$. In other words, we claim that b^G has defect group D if and only if $b^{\mathbf{N}_G(D)}$ has defect group D. If b^G has defect group D then so does b (for example, argue as in the first part of the Extended first main theorem) and by the same result, $b^{\mathbf{N}_G(D)}$ has defect group D. Conversely, if $b^{\mathbf{N}_G(D)}$ has defect group D then using that $(b^{\mathbf{N}_G(D)})^G = b^G$ we have that b^G has defect group D by Brauer's first main theorem.

Next we claim that we may assume b is G-invariant (now $D \lhd G$ so $D\mathbf{C}_G(D) \lhd G$). Now for any $D\mathbf{C}_G(D) \leqslant H \leqslant G$ we have that b^H is defined and covers b by Theorem 4.5 and Lemma 8.6. In particular both b^T and b^G cover b. Now $(b^T)^G$ is defined by the Fong–Reynolds theorem and $(b^T)^G = b^G$, so b^T is the Fong–Reynolds correspondent of b^G over b. Since the defect groups of b^T are defect groups of b^G, and $D \lhd G$ it follows that $\delta(b^T) = \{D\}$ if and only if $\delta(b^G) = \{D\}$, so we are done (we are using that if a defect group of a block is normal then it is the unique defect group of a block).

Therefore $D \lhd G$ and b is G-invariant. By Corollary 9.5, b^G is the unique block covering b. b^G satisfies the hypothesis on maximal defect in Fong's theorem 9.2 and we deduce that if $P \in \delta(b^G)$ we have $P \cap D\mathbf{C}_G(D) \in \delta(b) = \{D\}$, so $P\mathbf{C}_G(D)D = P\mathbf{C}_G(D)$ and $|G : P\mathbf{C}_G(D)|$ is not divisible by p. Since $P \cap \mathbf{C}_G(D)D = D$ it follows that $D < P$ if and only if $D\mathbf{C}_G(D) < P\mathbf{C}_G(D)$ (indeed, if $D\mathbf{C}_G(D) = P\mathbf{C}_G(D)$ then $P \subseteq D\mathbf{C}_G(D) \cap P = D$). Therefore p does not divide $|G : D\mathbf{C}_G(D)|$ if and only if $D = P$. □

9.3. Brauer's height zero conjecture

Stated in 1955 by Richard Brauer (and also included as part of Problem 23 of his famous list of problems), Brauer's height zero conjecture states that

Conjecture 9.7. *Let $B \in \mathrm{Bl}(G|D)$. Then $\mathrm{Irr}(B) = \mathrm{Irr}_0(B)$ if and only if D is abelian.*

We prove now the case where $D \lhd G$, a theorem due to W. F. Reynolds from 1963.

THEOREM 9.8 (Reynolds). *Let* $B \in \mathrm{Bl}(G|D)$ *and assume* $D \lhd G$. *Then* $\mathrm{Irr}(B) = \mathrm{Irr}_0(B)$ *if and only if* D *is abelian.*

PROOF. We argue by induction on $|G|$.

Let $b \in \mathrm{Bl}(D\mathbf{C}_G(D)|D)$ be a root of B and let T be its stablizer in G and B' the Fong–Reynolds correspondent of B in T. Since the defect groups of B' are defect groups of B, if $T < G$ then by induction, all characters of B' have height zero, and by the Fong–Reynolds correspondence, so do all characters of B. Thus we may assume b is G-invariant. In particular, it follows from Theorem 9.6 that $|G : D\mathbf{C}_G(D)|$ is p'.

Next we claim that the set of heights of characters in B coincides with the set of characters in b. Indeed, let $\chi \in \mathrm{Irr}(B)$. Since $b^G = B$ then we have that B covers b by Lemma 8.6. Now if $\chi \in \mathrm{Irr}(B)$ then any θ under χ lies in $\mathrm{Irr}(b)$. We have that $\chi(1)/\theta(1)$ divides $|G : D\mathbf{C}_G(D)|$, a p'-number. Thus $\chi(1)_p = \theta(1)_p$. Conversely if $\theta \in \mathrm{Irr}(b)$ we know there is $\chi \in \mathrm{Irr}(B)$ lying over θ, and arguing in the same way we get $\theta(1)_p = \chi(1)_p$. We conclude that

$$\{\chi(1)_p \mid \chi \in \mathrm{Irr}(B)\} = \{\theta(1)_p \mid \theta \in \mathrm{Irr}(b)\}$$

and the claim follows. In particular, $\mathrm{Irr}(b) = \mathrm{Irr}_0(b)$ if and only if $\mathrm{Irr}(B) = \mathrm{Irr}_0(B)$.

Now by the Corollary 8.12 of Theorem 8.11 we know that the set of heights of b is exactly $\{\nu(\xi(1)) \mid \xi \in \mathrm{Irr}(D)\}$. It follows that $\mathrm{Irr}(B) = \mathrm{Irr}_0(B)$ if and only if $\mathrm{Irr}(b) = \mathrm{Irr}_0(b)$ if and only if every character in $\mathrm{Irr}(D)$ is linear, which happens if and only if D is abelian. □

For the past 70 years, Brauer's height zero conjecture has been a central problem in the modular representation theory of finite groups. It was proved for p-solvable groups by D. Gluck and T. R. Wolf in 1984, in an already extremely complicated theorem involving group actions and orbit sizes. There is a whole book [**MW93**] devoted to the techniques involved in this proof (and related ones).

In 1988, T. Berger and R. Knörr proved that the "if" direction holds for every finite group provided that it holds for finite simple groups. This was finally proved by R. Kessar and G. Malle in 2013 in a widely celebrated paper.

In 2014, G. Navarro and B. Späth gave an extremely technical reduction theorem for the "only if" direction. We introduce some context:

CONJECTURE 9.9 (Alperin–McKay). *If* b *is the Brauer correspondent block of* B *then* $|\mathrm{Irr}_0(B)| = |\mathrm{Irr}_0(b)|$.

The Alperin–McKay conjecture was reduced to a problem on simple groups (known as the inductive Alperin–McKay conjecture) by B. Späth in 2011. It is one of the most important counting conjetures in our field. It was generalized to a much more general conjecture known as Dade's projective conjecture.

G. Robinson proved that, assuming Dade's projective conjecture, one could find certain bijections *above* height zero characters of Brauer correspondent blocks. M. Murai (who I believe was either an accountant or a high school teacher) proved that using this bijection, Brauer's height zero conjecture holds. The reduction theorem of Navarro and Späth states that, if the inductive Alperin–McKay conjecture holds for every finite simple group, then this bijection exists and therefore Brauer's height zero conjecture holds. In 2022 L. Ruhstorfer proved the inductive Alperin–McKay conjecture for $p = 2$, and as a corollary obtained Brauer's height zero conjecture for this prime (this paper was accepted recently in the Annals of Mathematics).

For odd primes the proof remained a challenge, as well as the proof of the inductive Alperin–McKay conjecture. Using a different reduction theorem and different conditions on finite simple groups, G. Malle, G. Navarro, A. A. Schaeffer Fry and P. H. Tiep were able to prove the "only if" direction for odd primes, thereby settling Brauer's height zero conjecture. This paper has also been accepted recently in the Annals (hopefully the graduate students reading this understand that this is an unbelievable achievement and out of reach for the vast majority of mathematicians).

Bibliography

[Bra64] R. Brauer. Some applications of the theory of blocks of characters of finite groups. I. *J. Algebra*, 1:152–167, 1964.

[Isa06] I. M. Isaacs. *Character theory of finite groups*. AMS Chelsea Publishing, Providence, RI, 2006. Corrected reprint of the 1976 original [Academic Press, New York; MR0460423].

[Isa08] I. M. Isaacs. *Finite group theory*, volume 92 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2008.

[MW93] O. Manz and T. R. Wolf. *Representations of solvable groups*, volume 185 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1993.

[Nav98] G. Navarro. *Characters and blocks of finite groups*, volume 250 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1998.

[Nav18] G. Navarro. *Character theory and the McKay conjecture*, volume 175 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2018.